PHY-715: Solid State Physics, UMass Amherst, Problem Set #3

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Due: Friday, April 2.

I. DECAYING WAVES

In class we showed that the dispersion relation of the harmonic chain is $\omega = \omega_{\max}|\sin\frac{ka}{2}|$, for which there is a maximum possible frequency of oscillation $\omega_{\max}$. What happens if a vibration with frequency $\omega > \omega_{\max}$ is forced upon the chain (say by a driving force)?

II. TRIATOMIC CHAIN

Consider a one dimensional mass-and-spring model with three different masses m_1, m_2, m_3 and three different springs (with constant $\kappa_1, \kappa_2, \kappa_3$) per unit cell. At $k = 0$, how many optical modes are there? Calculate the energies of these modes (for $k = 0$ only).

III. PHONON SPECTRUM IN 2D (ADAPTED FROM STEVE SIMON'S BOOK)

Consider a mass and spring model of a two-dimensional triangular lattice Λ with lattice spacing $a = 1$ as shown in the figure (assume the lattice is extended infinitely in all directions). Assume that identical masses m are attached to each of their six neighbors by equal springs of equal length and spring constant κ. Let the position of the sites in equilibrium be $\vec{r} \in \Lambda$, and let the small displacements from this equilibrium be $\vec{u}_{\vec{r}}$.

Assumes that the crystal starts in equilibrium with the springs unstretched, where all springs have unit length $a = 1$. Forces on the springs will be proportional to the amount of stretching. Note however, that if one of the masses is displaced in a direction perpendicular to one of its attaching springs, to linear order, the spring is not stretched at all (it is only rotated). Let us define the unit vectors labelling neighbors $\vec{a}_1 = (1, 0)$, $\vec{a}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $\vec{a}_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. The total elastic potential energy is:

$$U = \frac{\kappa}{2} \sum_{\vec{r}\in\Lambda} \sum_{j=1}^{3} \left(\left(\vec{u}_{\vec{r}} - \vec{u}_{\vec{r}+\vec{a}_j} \right) . \vec{a}_j \right)^2 , \tag{1}$$

where here we have included each spring once in the sum and we have dotted the displacement with its direction so as to only count stretching of the spring and not rotation. Write down the equations of motion for the displacements $\vec{u}_{\vec{r}}$, and look for solutions of the form $\vec{u}_{\vec{r}} = \vec{u} e^{i\omega t - i\vec{k}.\vec{r}}$. Show that the phonon spectrum is determined by the eigenvalue equation $\hat{D}(\vec{k})\vec{u} = \omega^2 \vec{u}$ where $\hat{D}(\vec{k})$ is a 2×2 matrix that you will determine. Find the corresponding eigenvalues and plot/visualize the dispersion relations $\omega_\pm(\vec{k})$ using the software of your choice.

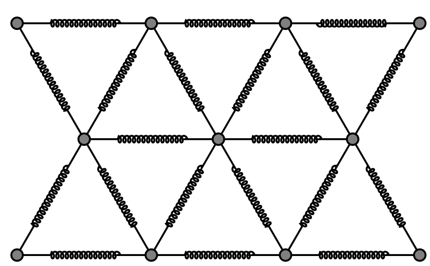

FIG. 1: Phonons on the triangular lattice. (figure from Steve Simon's book)