PHY-421: Mechanics, UMass Amherst, Problem Set #7

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Due: Friday, Oct 23. (Late homework receives 50% credit.)

I. SHORTEST PATH BETWEEN TWO POINTS

Consider a two-dimensional space with coordinates (x, y), and a function $y = y(x)$ that defines a curve connecting two points $A = (x_A, y_A = y(x_A))$ and $B = (x_B, y_B = y(x_B))$. Write the length of this curve as a functional of $y(x)$, and minimize this functional to show that the shortest path between A and B is a straight line.

II. CONSERVATION LAW

Consider the action $S = \int dx L(y(x), y'(x))$. In class we showed that the action is stationary $\delta S = 0$ for $y(x)$ satisfying the Euler-Lagrange equations

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}. \tag{1}$$

Show that if the Lagragian L doesn't depend explicitly on x (that is, if $\partial L/\partial x = 0$), then

$$\frac{d}{dx}\left(L - \frac{\partial L}{\partial y'}y'\right) = 0. \tag{2}$$

III. SOAP FILM (NOT GRADED, HONORS COLLOQUIUM)

Consider a soap film between two coaxial circular rings of the same radius r. We denote by x the coordinate of the central axis. The circular rings are positioned at $x = -d/2$ and $x = +d/2$ respectively. By symmetry arguments, neglecting the gravity, the film corresponds to a surface of revolution along the x axis with radius $y(x)$, with $y(-d/2) = y(+d/2) = r$. At the thermodynamic equilibrium, we will admit that the soap film minimizes its surface.

1. Argue that the area of the soap film is given by

$$A[y] = \int_{-d/2}^{d/2} dx 2\pi y(x)\sqrt{1 + (y'(x))^2}. \tag{3}$$

2. Use the results of Problem II to show that the equation of the soap film satisfies the differential equation

$$\frac{y}{\sqrt{1 + (y')^2}} = C, \text{with } C \text{ a constant.} \tag{4}$$

3. Solve this equation by separation of variables, and find the shape of the soap film $y(x)$.

Hint: Use the integral:

$$\int \frac{dy}{\sqrt{y^2 - 1}} = \text{ArcCosh } y + \text{constant},$$

where ArcCosh is the inverse of the hyperbolic cosine function Cosh.