PHY-817: Advanced Statistical Physics, UMass Amherst, Problem Set #4

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Due: Friday April 10.

I. COUPLED ISING MODELS AND EMERGENT SYMMETRY

Let us consider two coupled Ising models in three dimensions with spin variables S_i^1, and S_i^2, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The symmetry group corresponds to two independent spin flip symmetries $S_i^1 \to -S_i^1$ and $S_i^2 \to -S_i^2$. We also assume a symmetry that exchanges the two models $1 \leftrightarrow 2$, and consider the Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<i,j>} \left(S_i^1 S_j^1 + S_i^2 S_j^2\right) + g \sum_{<i,j>} S_i^1 S_j^1 S_i^2 S_j^2.$$

The goal of this problem is to show that the critical point separating the ordered and disordered phases in this model has an emergent $O(2)$ symmetry (much larger than the microscopic symmetry group!), and is in the universality class of the $O(2)$ model.

1. Use symmetry arguments to carefully explain why this problem can be described by the following Landau-Ginzburg action:

$$S = \int d^d r \left[\frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}(\nabla\phi_2)^2 + \frac{t}{a^2}(\phi_1^2 + \phi_2^2) + \frac{u}{a^{4-d}}(\phi_1^4 + \phi_2^4) + \frac{v}{a^{4-d}}\phi_1^2\phi_2^2\right],$$

with a a UV cutoff.

2. Since the upper critical dimension of the Ising model is $d_c = 4$, we can study the RG flows of this model perturbatively for small $u, v \ll 1$ in dimension $d = 4 - \epsilon$. We will ignore the RG equation for t as it will simply act as a relevant thermal perturbation that drives the phase transition. We therefore focus on the flow on the $t = 0 + \mathcal{O}(\epsilon^2)$ manifold to identify the fixed point that controls the critical behavior of this model. By computing the relevant operator product expansions at the Gaussian fixed point, derive the RG flow equations for u and v

$$\frac{du}{d\ell} = \epsilon u - 72u^2 - 2v^2 + \dots$$
$$\frac{dv}{d\ell} = \epsilon v - 16v^2 - 48uv + \dots$$

Note that as in class, we have assumed normal ordering of the operators at the Gaussian fixed point and we've absorbed unimportant angular prefactors in the definition of the couplings.

3. Identify the fixed points of these flow equations. In addition to the Gaussian fixed point and to the Wilson-Fisher fixed point corresponding to decoupled Ising models with $(u^\star, v^\star) = (\mathcal{O}(\epsilon), 0)$, you should find two new fixed points where $u^\star, v^\star$ are of order ϵ.

4. Study the stability of these fixed points by computing their RG eigenvalues (you can use Mathematica if you want). In particular, show that the perturbation $v\phi_1^2\phi_2^2$ is relevant at the decoupled Ising models fixed point, and that the (stable) fixed point controlling the critical behavior of the coupled Ising models has $v^\star = 2u^\star$. Explain why this means that there is an emergent $O(2)$ rotation symmetry at this critical point.

II. TRICRITICAL POINT IN THE ISING MODEL AND ϕ^6 THEORY

Let us consider the critical behavior of a tricritical point described by a ϕ^6 theory

$$S = \int d^d r \left[\frac{1}{2}(\nabla\phi)^2 + \frac{t}{a^2}\phi^2 + \frac{u}{a^{4-d}}\phi^4 + \frac{g}{a^{2(3-d)}}\phi^6\right],$$

with a a UV cutoff, and where $t = 0$ and $u = 0$ at the tricritical point. Recall that by studying the Gaussian fluctuations around the saddle point, you've shown in the problem set #2 that the upper critical dimension of this model is $d_c = 3$.

1. What is the scaling dimension of ϕ^6 at the Gaussian fixed point? Justify the powers of a in the action.
2. Examine the RG flow equations of this model near the tricritical point below three dimensions using an $\epsilon = 3 - d$ expansion. Show that there is a non-trivial RG fixed point with $g^\star = \mathcal{O}(\epsilon)$ and $u^\star = t^\star = 0 + \mathcal{O}(\epsilon^2)$ controlling the tricritical behavior.
3. Show that there are two relevant scaling variables t and u at the tricritical point, with two distinct scaling dimensions. When $t \neq 0$ or $u \neq 0$, the correlation length scales as

$$\xi \sim t^{-\nu_t}, \quad \xi \sim u^{-\nu_u},$$

respectively. Calculate the critical exponents ν_t and ν_u below three dimensions to first order in $\epsilon = 3 - d$.