PHY-817: Advanced Statistical Physics, UMass Amherst, Problem Set #2

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Due: Friday, Feb 28 by 5pm.

I. GAUSSIAN INTEGRALS AND WICK'S THEOREM

Consider n real variables ϕ_a drawn from a Gaussian ensemble such that

$$\langle \mathcal{O}(\vec{\phi}) \rangle = \frac{1}{Z} \int \prod_{i=1}^{n} d\phi_a \mathcal{O}(\vec{\phi}) \mathrm{e}^{-\frac{1}{2}\vec{\phi}^T M \vec{\phi}},$$

with M a real symmetric matrix.

1. Compute the normalization factor Z defined by $\langle 1 \rangle = 1$.
2. Show that $\langle \mathrm{e}^{\sum_a \gamma_a \phi_a} \rangle = \mathrm{e}^{\frac{1}{2}\sum_{a,b} \gamma_a \gamma_b (M^{-1})_{ab}}$
3. By Taylor expanding (or differentiating) both sides, compute $\langle \phi_a \rangle$, $\langle \phi_a \phi_b \rangle$, $\langle \phi_a \phi_b \phi_c \rangle$ and $\langle \phi_a \phi_b \phi_c \phi_d \rangle$.

II. MULTICRITICAL POINT AND FLUCTUATIONS

Consider a system with order parameter m and symmetry $m \to -m$. We are interested in multicritical points where the Landau expansion of the free energy takes the form

$$F = \int d^d x \left(\frac{K}{2} (\nabla m)^2 + a_0 (T - T_c) m^2 + b m^{2n} \right),$$

with $T = T_c$ at the multicritical point. Here, $n \geq 2$ is an integer and $a_0, K, b > 0$.

1. Imagine tuning the temperature T across the multicritical point. What is the (mean-field) magnetization exponent β as a function of n?
2. Substitute this back into the free energy to compute the specific heat critical exponent α.
3. Compare this with the contribution to the free energy from the Gaussian fluctuations around the saddle point (mean-field) solution. Show that the mean field contribution to the free energy dominates at the critical point provided $d > d_c$ where you will determine the upper-critical dimension d_c as a function of n.

III. SUPERFLUID HE^4-HE^3 MIXTURES

The superfluid He^4 order parameter is a complex number $\psi(\mathbf{x})$, where $\langle |\psi|^2 \rangle \neq 0$ indicates a superfluid phase. In the presence of a concentration $\phi(x)$ of He^3 impurities, the system has the following Landau-Ginzburg energy

$$\beta \mathcal{H}[\psi, \phi] = \int d^d x \left(\frac{K}{2} |\nabla \psi|^2 + t |\psi|^2 + u |\psi|^4 + v |\psi|^6 + \frac{\phi^2}{2\sigma^2} - \gamma \phi |\psi|^2 \right),$$

with K, u and v positive.

1. Integrate out the He^3 concentrations to find the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}[\psi]$ for the superfluid order parameter, given by

$$Z = \int \mathcal{D}\psi \ \mathrm{e}^{-\beta \mathcal{H}_{\mathrm{eff}}[\psi]} = \int \mathcal{D}\psi \mathcal{D}\phi \ \mathrm{e}^{-\beta \mathcal{H}[\psi, \phi]}.$$

2. Obtain the phase diagram for $\beta \mathcal{H}_{\mathrm{eff}}[\psi]$ using a saddle point approximation. Show that there is a line of second order transitions which joins a line of first order transitions at a special point, called a tricritical point (see problem II).