# Random Systems

- Real systems are usually "dirty": many impurities, etc.
- Model this using random couplings (= disorder)

Random Ising model: $\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j$

$J_{ij}$ random variables drawn from, say, a Gaussian distribution.

- Disordered systems have a very rich phenomenology: spin glasses, etc.

Here, we focus on a simple question:

> Does disordered / randomness modify the critical behavior (universality class) of a transition?

→ is weak disorder relevant at a given RG fixed point?

## (A) Replica Trick

Important to distinguish:

* "Annealed" disorder: impurities are dynamical variables like the spins in our Ising model, and should be averaged over in the partition function: $Z_{\text{annealed}} = \overline{Z}$ ← average over $J_{ij}$

* "Quenched" Disorder: impurities are "frozen", $Z$ depends on $J_{ij}$
→ loose translation invariance.

However, for a thermodynamic system: $\frac{F}{N} \simeq \frac{\overline{F}}{N}$ (we say that the free energy is self averaging)

This is because $F = \sum_{\substack{\text{subsystems} \\ \{i\}}} F_i$ in the thermodynamic limit.

$F/N = \left( \sum_{\text{subsystem}} F/N_{\text{subsystem}} \right) / \#\text{ subsystems}$

- Consider quenched disorder: we need to average $F$: compute $\overline{F}$ ← over disorder

→ as usual, all physical quantities follow from taking derivatives.

- **Problem**: computing $\frac{\overline{F}}{T} = -\overline{\log Z}$ is hard!

**Replica trick**: use formula: $\boxed{\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}}$

$\Rightarrow Z^n$ = partition function of $n$ copies ("replicas") of the system for $n \in \mathbb{N}$

and $\overline{Z^n}$ usually much easier to compute! (for $n$ integer)

$\Rightarrow$ then analytically continue to $n \to 0$ (can be subtle! may not commute with thermodynamic limit!)

## (B) Harris Criterion

Consider a clean model: $\beta\mathcal{H} = \beta\mathcal{H}^* + t\int d^d x\, \phi_t(x)$

RG fixed point

thermal perturbation $\Delta_t = d - \nu^{-1}$

**Coarse graining**: impurities couple to energy density $\varepsilon(x) \sim \phi_t(x) + \ldots$

$$\Rightarrow \boxed{\beta\mathcal{H} = \beta\mathcal{H}^* + \int d^d x\, t(x)\, \phi_t(x) + \ldots}$$

disorder = random variables not correlated in space: $\overline{\delta t(x)\delta t(x')} = \sigma_t^2\,\delta(x-x')$ with $\delta t(x) = t(x) - \overline{t(x)}$

**Replica trick**: $\overline{Z^n} = \int D\vec{\phi}\ \overline{e^{-\sum_{a=1}^{n}\beta\mathcal{H}[\phi_a]}}$

degrees of freedom in the model $= \phi$

$\vec{\phi} = (\phi_1, \ldots, \phi_n)$ in replicated theory

$$\Rightarrow \overline{Z^n} = \int D\vec{\phi}\, e^{-\beta\sum_{a=1}^{n}\mathcal{H}^*[\phi_a]}\ e^{-\overline{t}\int d^d x \sum_a \phi_t^a + \frac{\sigma_t^2}{2}\int d^d x \sum_{a b}\phi_t^a(x)\,\phi_t^b(x) + \ldots}$$

cumulant expansion

disorder strength

used $\overline{\delta t(x)\delta t(x')} = \sigma_t^2 \delta$

* $1^{st}$ term = trivial: $\bar{F}$ just shifts $T_c$

* $2^{nd}$ term: $\sigma_t^2 = \overline{F^2} - (\bar{F})^2$: Variance of the disorder (= 0 if clean system)

$\Rightarrow$ effective interaction between replicas: $-\frac{\sigma_t^2}{2}\int d^d r \sum_{a,b=1}^{n} \phi_t^a(r)\, \phi_t^b(r)$

$\Rightarrow$ <u>$a = b$ terms</u>: same replica, $(\phi_t^a)^2$ not a scaling operator in general

but $(\phi_t^a)^2 \simeq c\, \phi_t^a + \dots \Rightarrow$ effective shift in $T_c$, doesn't change critical exponents.

$\Rightarrow$ <u>$a \neq b$ terms</u>: $\mathcal{O}(r) = \sum_{a \neq b} \phi_t^a(r)\, \phi_t^b(r)$ relevant? Scaling dimension?

$$S_0 = \beta \sum_{a=1}^{n} \mathcal{H}^*[\phi_a] \quad \text{decoupled replicas}$$

$$\Rightarrow \langle \mathcal{O}(\vec{r})\, \mathcal{O}(\vec{r}\,')\rangle_0 = 2n(n-1) \langle \phi_t^a(\vec{r})\, \phi_t^a(\vec{r}\,')\rangle_0^2 \sim \frac{2n(n-1)}{|\vec{r} - \vec{r}\,'|^{4\Delta_t}}$$

this yields: $\Delta_{\mathcal{O}} = 2\Delta_t$

$y_{\mathcal{O}} = d - 2\Delta_t = 2/\nu - d$

(Weak) disorder is irrelevant if $y_{\mathcal{O}} < 0 \iff \boxed{d\nu > 2}$ Harris Criterion

correlation length exponent of the clean fixed point

- if this inequality is satisfied: disorder irrelevant, and just shifts $T_c$ (but does not change critical exponents).
- if $d\nu < 2$: new fixed point / physics in the presence of disorder