Non-linear Sigma Model

(A) Interactions between Goldstone modes : non-linear σ-model

Consider $O(n)$ model : recall that if $d \leqslant 2$ (and $n \geqslant 2$), Goldstone modes destroy long range order.
↳ continuous spins

- $d > 2$: ordinary transition $\Rightarrow$ suggests low T and $d = 2 + \varepsilon$ expansion

$$-\beta \mathcal{H} = K \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = K \sum_{\langle i,j \rangle} \left(1 - \frac{(\vec{S}_i - \vec{S}_j)^2}{2} \right) \quad \text{since } \vec{S}_i^{\,2} = 1$$

$\Rightarrow$ continuum limit :
$$\boxed{\beta \mathcal{H} = \frac{K}{2} \int \frac{d^d r}{a^{d-2}} (\nabla \vec{S})^2} \quad \text{with } \vec{S}^2(\vec{x}) = 1$$
$\rightarrow$ (+ constant)

Not a Gaussian theory because of constraint $\vec{S}^2 =$

- Set $K = \frac{1}{T}$ with T temperature ($K_B = J = 1$). $\underbrace{T = 0 \text{ groundstate}}_{\infty\text{-many possibilities}}$ $\vec{S} = \underbrace{(1, 0, 0, \ldots)}_{n \text{ components}}$

Set $\vec{S} = \Big(-\sqrt{1 - \vec{\sigma}^2}, \underbrace{\sigma_1, \ldots, \sigma_{n-1}}_{\substack{n-1 \text{ transverse fluctuations} \\ \Rightarrow \text{ Goldstone modes}}}\Big)$: $\quad \beta \mathcal{H} = \frac{1}{2T} \int \frac{d^d r}{a^{d-2}} \left[(\nabla \vec{\sigma})^2 + (\nabla \sqrt{1 - \vec{\sigma}^2})^2 \right]$

- Note : we are ignoring the Jacobian $\mathcal{D}\vec{S}\, \delta(\vec{S}^2 - 1) \longrightarrow \mathcal{D}\vec{\sigma}$ but it is not important

Let us denote $\beta \mathcal{H} = S$ (action) and rescale $\sigma \longrightarrow \sqrt{T}\, \sigma$ and $T \ll 1$

$$\boxed{S = \frac{1}{2} \int \frac{d^d r}{a^{d-2}} \left[(\nabla \vec{\sigma})^2 + T\, (\vec{\sigma} \cdot \nabla \vec{\sigma})^2 + \ldots \right]}$$

Non-linear σ-model
interactions between Goldst…

Ⓑ RG analysis in $d=2$: asymptotic freedom

. In principle, we could use the OPE approach to derive the perturbative RG flow of this problem, but it is quite tricky here...

. Focus first on $d=2$, and use our knowledge of $n=2$: $\frac{dK}{d\ell}=0$
(ignoring non-perturbative aspects like vortices)

Rewrite : $\vec{S} = \left(\sqrt{1-\vec{\pi}^2} \cos\theta, \sqrt{1-\vec{\pi}^2} \sin\theta, \vec{\pi} \right)$

$\vec{\pi}$: $n-2$ components

$$\Rightarrow \quad S = \frac{K}{2} \int d^2x \left[\underbrace{(\nabla\theta)^2}_{\text{"slow" degree of freedom}} (1-\vec{\pi}^2) + \left(\nabla\sqrt{1-\vec{\pi}^2}\right)^2 + (\nabla\vec{\pi})^2 \right]$$

$\vec{\pi}$ small (imagine $\pi \to \sqrt{T}\pi$)

$$\simeq \frac{1}{2T} \int d^2x \left[(\nabla\theta)^2 \left(1 - \underbrace{\vec{\pi}^2\right)}_{V} + (\nabla\vec{\pi})^2 + \mathcal{O}(T^2) \right]$$

Integrate out $\vec{\pi}$ ("partial trace") to get effective dynamics of θ:

$$e^{-S_{eff}[\theta]} = \underbrace{\int \mathcal{D}\vec{\pi}(\vec{x})\, e^{-S[\theta,\vec{\pi}]}}_{\propto \langle e^{-V} \rangle^{\circ}_{\vec{\pi}}}$$

Can be done order by order in T

$Z^{\circ}_{\pi} = \int \mathcal{D}\vec{\pi}\, e^{-\frac{1}{2T}\int d^2x (\nabla\vec{\pi})^2}$ independent of θ

$$\simeq e^{-\frac{1}{2T}\int d^2x (\nabla\theta)^2} \, e^{+\frac{1}{2T}\int d^2x (\nabla\theta)^2 \langle \vec{\pi}^2 \rangle^{\circ}_{\vec{\pi}}}$$

up to irrelevant prefactors

To leading order: $\langle e^{-V} \rangle^{\circ}_{\pi} \simeq e^{-\langle V \rangle^{\circ}_{\pi}}$, we can replace $\vec{\pi}^2(\vec{x}) \to \langle \vec{\pi}^2 \rangle$

. $\langle \pi_\alpha(x) \pi_\beta(y) \rangle_0 = T \delta_{\alpha\beta} \int \frac{d^2k}{(2\pi)^2} \frac{e^{-i\vec{k}.\vec{n}}}{k^2} \Rightarrow \langle \vec{\pi}^2 \rangle = \frac{(n-2)T}{2\pi} \int_0^{\Lambda_{UV} \sim 1/a} \frac{dK}{K}$

. θ described by XY model with : $\boxed{K_{eff} = K - \frac{(n-2)}{2\pi} \int^{\Lambda} \frac{dK}{K}}$

or $T_{eff} \simeq T + T^2 \frac{n-2}{2\pi} \underbrace{\int_0^{\Lambda_{UV}} \frac{dk}{k}}_{\text{divergent}} + \ldots$

divergent at low energy, perturbation theory breaks down

cf what we did for the XY model: $\int_0^{\Lambda} = \underbrace{\int_0^{\Lambda/b}}_{} + \int_{\Lambda/b}^{\Lambda}$

change variables to restore $\Lambda/b \to \Lambda$

$k' = kb$

$$\Rightarrow T_{eff} = T + T^2 \frac{n-2}{2\pi} \int_0^{\Lambda} \frac{dk'}{k'} + T^2 \frac{n-2}{2\pi} \log b + \mathcal{O}(T^3)$$

$$= \tilde{T} + \tilde{T}^2 \frac{n-2}{2\pi} \int_0^{\Lambda} \frac{dk}{k} \quad \text{with } \tilde{T} = T + T^2 \frac{n-2}{2\pi} \log b + \mathcal{O}(T^3)$$

↳ momentum shell RG: $\tilde{T}$ = effective coupling by integrating modes of $\vec{\pi}$ between Λ/b and Λ

$\Rightarrow$ change in UV cutoff can be absorbed in redefinition $T \longrightarrow \tilde{T}$

take $b = e^{\delta \ell} \simeq 1 + \delta \ell$, $\frac{dT}{d\ell} = \frac{\tilde{T} - T}{\delta \ell}$ $\Big\} \Rightarrow$ $\boxed{\frac{dT}{d\ell} = T^2 \frac{n-2}{2\pi} + \ldots}$

- flows to strong coupling at low energy (IR)
- Free at high energy

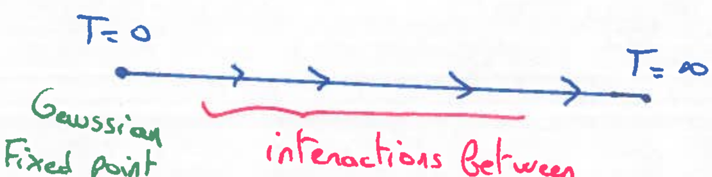
"Asymptotic freedom" → not relevant for us!

- We see that $n = 2$ is special ($K = 1/T$ marginal): only 1 Goldstone mode here, no interaction

↳ cf. our discussion of XY

- If $n > 2$, T increases under RG: perturbation theory breaks down

$\Rightarrow$ natural guess: disordered phase



interactions between $n-1$ Goldstone modes leads to a disordered state

Correlation length:

$$\xi(T) = \underbrace{e^{\ell}}_{b} \xi(T(\ell))$$

ℓ^* so that $\xi(T(\ell)) = \mathcal{O}(1)$ and $T(\ell) \simeq \mathcal{O}(1)$

$$T^{-1}(\ell) = T^{-1} - \frac{n-2}{2\pi} \ell + \ldots$$

$$\Rightarrow \boxed{\xi(T) \sim a\, e^{2\pi/(n-2)T}}$$ as $T \to 0$

Self-generated lengthscale "Dimensional transmutation"

(C) $O(n)$ model in $d = 2+\varepsilon$ dimensions

- In $d > 2$, we expect a normal ordering transition.

- $$S = \frac{1}{2T}\int \frac{d^d x}{a^{d-2}} (\nabla \vec{s})^2 \Rightarrow [(\nabla \vec{s})^2] = 2 = \Delta_{(\nabla s)^2} \Rightarrow \frac{dT}{d\ell} = -\varepsilon T + \dots$$

  $\frac{d^d x}{a^{d-2}}$: dimensionless

  $a \to a\ell$, $T' = \ell^{2-d}\, T$

  ($T \to T'$ to keep S invariant)

  eigenvalue at the $T=0$ fixed point: $-\varepsilon$

- To leading order in ε, doesn't affect $O(T^2)$ term: $\boxed{\frac{dT}{d\ell} = -\varepsilon T + \frac{n-2}{2\pi} T^2 + \dots}$

  $\dots$: $O(\varepsilon T^2)$, $O(T^3)$

symmetry breaking: $O(n) \to O(n-1)$
+ $n-1$ Goldstone modes

linearize near T^*
$$\frac{dT}{d\ell} = \varepsilon (T - T^*) + \dots$$

$y_t = \varepsilon \Rightarrow \boxed{\nu = \frac{1}{\varepsilon} + O(1)}$

here $\varepsilon = d-2$

- Magnetization exponent: for $\varepsilon > 0$, T irrelevant near $T=0$ fixed point and we can use perturbation theory. Magnetization $= m = \langle \sqrt{1-\sigma^2} \rangle \simeq 1 - \frac{\langle \sigma^2 \rangle}{2} + \dots$ in the ordered phase. We have:
$$\langle \vec{\sigma}^2 \rangle = (n-1) T \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = (n-1) T \frac{S_d}{(2\pi)^d} \underbrace{\int_0^\Lambda dk\, k^{d-3}}_{\frac{\Lambda^\varepsilon}{\varepsilon}}$$

$\Rightarrow$ leading order in ε: $\langle \vec{\sigma}^2 \rangle = \frac{T(n-1)}{2\pi\varepsilon} \Rightarrow m = 1 - T\frac{(n-1)}{4\pi\varepsilon} + \dots$ (nice and finite)

Now extrapolating this result near T^*, we expect $m \sim (T^* - T)^\beta \sim 1 - \beta\, T/T^*$ at small T $\Rightarrow \beta / \left(\frac{2\pi\,\varepsilon}{n-2}\right) = \frac{n-1}{4\pi\varepsilon} \Rightarrow$
$$\boxed{\beta = \frac{n-1}{2(n-2)} + O(\varepsilon)}$$