A Prelude to Quantum Field Theory
Problems for Chapter 3

1. **Euler-Lagrange, with second derivatives.** Using eq. (3.30) right number?, show that the equation of motion following from the Lagrangian $L = -\frac{1}{2} \phi [\Box + m^2] \phi$ is the same as for $L = \frac{1}{2} [\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2]$. In the action, these two forms are related by integration by parts when one drops the surface contribution.

2. **Photon stress-energy.** Derive eq. (3.81) starting directly from the Lagrangian and the definition of photon field. There are interesting cancellations that allow to get to the right result.

3. **The Landau theory of phase transitions as a field theory.** Let us consider a one-component ferromagnet described by a field $\phi(t, x)$ representing the local magnetization. At equilibrium, the field is time-independent, so that the free energy will be given by an expression of the form

$$E = \frac{1}{2} (\nabla \phi)^2 + V(\phi),$$

where we assume

$$V(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda_4}{4} \phi^4 + \frac{\lambda_6}{6} \phi^6,$$

where $\lambda_6 > 0$ is needed to guarantee stability of the system. In general the parameters of this theory depend on temperature. In particular, in the Landau theory of ferromagnetism one assumes $m^2 = a(T - T_0)$ with $a$ and $T_0$ positive constants and one neglects the dependence of $\lambda_4$ and $\lambda_6$ on temperature.

(a) Use variational techniques to derive the equation that determines the configuration of minimum free energy.

(b) Assume that $\lambda_4 < 0$ and $\lambda_6 > 0$. Plot the energy of the vacuum state as a function of $\phi$ for various values of the temperature. Show that at large enough temperatures the energy is minimal at $\phi = 0$, but there is a temperature $T_c > T_0$ where the system displays nonzero magnetization $\phi \neq 0$. Compute $T_c$. Sketch a
plot of the value of $\phi$ that minimizes energy as a function of $T$. Is this a first or a second order phase transition? [In a first order phase transition the order parameter changes discontinuously as a function of $T$; in a second order phase transition the first derivative is discontinuous]

(c) Repeat the steps of question (b) above in the case $\lambda_4 > 0$. At what temperature does the phase transition happen in this case? Is it a first or a second order phase transition?

4. A theory with higher derivatives. Show that a theory with a higher derivative with a Lagrangian given by

$$\mathcal{L} = \frac{\sigma}{\Lambda^2} (\partial_\mu \partial^\mu \phi) (\partial_\nu \partial^\nu \phi),$$

actually describes two scalar fields, and at least one of them has a wrong sign kinetic term. In the Lagrangian above $\sigma$ can be either +1 or -1 and $\Lambda$ is a constant with the dimensions of an energy.

To do this, rewrite $(\partial_\mu \partial^\mu \phi) (\partial_\nu \partial^\nu \phi)$ as $\chi^2$, where $\chi$ is an auxiliary field that is imposed to be equal to $(\partial_\mu \partial^\mu \phi)$ by the addition to $\mathcal{L}$ of a Lagrange multiplier $\lambda(\chi - \partial_\mu \partial^\mu \phi)$. Now the Lagrangian has three fields $\phi$, $\chi$ and $\lambda$. Eliminate all the non dynamical fields from the Lagrangian and diagonalize the kinetic term. You should obtain, irrespective of the sign of $\sigma$, one field with positive and one field with negative coefficient of the kinetic term. What are the masses of these two fields?

5. Charge algebra. Consider a theory with $N$ real scalar fields whose Lagrangian in symmetric under the transformation

$$\phi_i(x^\mu) \rightarrow \sum_{j=1}^{N} T_{ij} \phi_j(x^\mu),$$

where the matrix $T$ is an element of a Lie group of dimension $M$, that means that the generic matrix $T$ can be written as $T = e^{i \sum_k \alpha_k \tau_k}$, where the $\alpha_k$s are real numbers and the $\tau_k$ are the generators of the group, i.e. $N \times N$ matrices that satisfy some algebra

$$[\tau_k, \tau_l] = i \sum_m C_{klm} \tau_m.$$
where \([..., ...]\) denotes the commutator and where the constants \(C_{klm}(= -C_{lkm})\) are known as the structure constants of the Lie group.

(a) Compute the expression of Nöther currents \(j^\mu_k\) associated to the invariance of the Lagrangian under variations with respect to the parameter \(\alpha_k\).

(b) If \(\hat{Q}_k\) is the charge operator associated to \(j^\mu_k\), show that the charges satisfy the same Lie algebra as the generators of the group that is associated to those charges. This means that

\[
[\hat{Q}_k, \hat{Q}_l] = i \sum_{m} C_{klm} \hat{Q}_m. \tag{1}
\]

[Note that you might have to judiciously choose the factors of the imaginary unit \(i\) in the definition of \(j^\mu_k\) in order for equation (1) above to be valid.]