Math 421 **Problem Set 1 Answers** **Fall, 2010**

1. **[20 pt]** This part of Homework 1 consists of the five problems from *WeBWorK* set 421HW1.

2. (a) [Page 6 # 5 (b)] **[10 pt]** From the del Ferro-Tartaglia formula, one solution of this depressed cubic is:

$$x = \sqrt[3]{65 + 142i} + \sqrt[3]{65 - 142i} \qquad \textbf{[2 pt]}$$

Following Bombelli's method, assume this is real and that, in fact, there are real u, v with

$$\sqrt[3]{65 + 142i} = u + iv, \ \sqrt[3]{65 - 142i} = u - iv.$$

Then $(u + iv)^3 = 65 + 142i$. Cubing and simplifying gives:

$$(u^3 - 3uv^2) + (3u^2v - v^3)i = 65 + 142i \qquad \textbf{[2 pt]}$$

Equating real and imaginary parts gives:

$$\begin{cases} u^3 - 3uv^2 = 65 \\ 3u^2v - v^3 = 142 \end{cases} \qquad \textbf{[1 pt]}$$

Factor the left sides to obtain:

$$\begin{cases} u(u^2 - 3v^2) = 65 \\ v(3u^2 - v^2) = 142 \end{cases} \qquad \textbf{[1 pt]} \qquad (*)$$

Now seek solutions u, v of this system that are **positive integers**. Since 65 has prime factorization $65 = 5 \cdot 13$, two possible solutions of the first equation are

$$\big(u = 5 \text{ and } u^2 - 3v^2 = 13\big) \text{ or } \big(u = 13 \text{ and } u^2 - 3v^2 = 5\big)$$

Now $u = 5$ and $u^2 - 3v^2 = 13$ means

$$u = 5 \text{ and } v = 2. \qquad \textbf{[1 pt]} \qquad (*)$$

And those values of u, v also satisfy the second equation in (*). [You could, instead, run through the other possibility and see where it leads, or start with the second equation in (*).]

Thus one solution to the original cubic is

$$x = (u + iv) + (u - iv) = 2u = 2(5) = 10. \qquad \textbf{[1 pt]}$$

You may now directly check that $x = 10$ does indeed satisfy the equation $x^3 - 87x - 130 = 0$.

Divide $x^3 - 87x - 130$ by $x - 10$ (by hand or by using *Mathematica*'s `PolynomialQuotient` to obtain

$$x^3 - 87x - 130 = (x - 10)(x^2 + 10x + 13).$$

From the quadratic formula, the zeros of the quadratic factor are $x = -5 \pm 2\sqrt{3}$. Thus the zeros of the original cubic are:

$$\boxed{x = 10, \quad -5 + 2\sqrt{3}, \quad -5 - 2\sqrt{3}} \qquad \textbf{[2 pt]}$$

(b) [Page 6 #6 (a)] **[10 pt]** The given cubic $z^3-6z^2-3z+18$ has the form $z^3+a_2z^2+a_1z+a_0$ with $a_2=-6$. Then the Cardan substitution to be used is

$$z = x - a_2/3 = x - (-6)/3 = x + 2. \qquad \textbf{[1 pt]}$$

In terms of the new variable x, the original cubic becomes

$$\begin{aligned}(x+2)^3 - 6(x+2)^2 - 3(x+2) + 18 &= (x^3+6x^2+12x+8) - 6(x^2+4x+4) - 3(x+2) + 18 \\ &= x^3 - 15x - 4. \qquad \textbf{[3 pt]}\end{aligned}$$

This is the very same depressed cubic analyzed in the text; as shown there, the delFerro-Tartaglia formula gives as one solution

$$x = 4. \qquad \textbf{[2 pt]}$$

Long division gives

$$x^3 - 15x - 4 = (x-4)(x^2+4x+1).$$

By the quadratic formula the solutions of x^2+4x+1 are $x=-2\pm\sqrt{3}$. Thus the three solutions of the depressed cubic are

$$x = 4, \quad -2+\sqrt{3}, \quad -2-\sqrt{3}. \qquad \textbf{[2 pt]}$$

To find the solutions of the original cubic (with variable z) from the solutions of the depressed cubic (with variable x) use the relation $z=x+2$ to obtain

$$\boxed{z = 6, \quad \sqrt{3}, \quad -\sqrt{3}.} \qquad \textbf{[2 pt]}$$

3. [Verify distributive law from operations definitions in terms of ordered pairs] **[20 pt]** Let $z=(a,b), w=(u,v), \zeta=(s,t)$. Then:

$$\begin{aligned}z(w+\zeta) &= (a,b)(u+s, v+t) \\ &= \big(a(u+s) - b(v+t), a(v+t) + b(u+s)\big) \\ &= (au + as - bv - bt, av + +at + bu + bs) \qquad \textbf{[10 pt]}\end{aligned}$$

On the other hand,

$$\begin{aligned}zw + z\zeta &= (au - bv, av + bu) + (as - bt, at + bs) \\ &= (au - bv + as - bt, av + bu + at + bs), \qquad \textbf{[10 pt]}\end{aligned}$$

which is the same ordered pair as the value of $z(w+\zeta)$.

4. [Page 15 #6] Write $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ with x_1, y_1, x_2, y_2 real.

(a) **[10 pt]** This is **true** because:

$$\operatorname{Re}(z_1+z_2) = \operatorname{Re}\big((x_1+x_2) + i(y_1+y_2)\big) = x_1 + x_2 = \operatorname{Re} z_1 + \operatorname{Re} z_1$$

(d) **[10 pt]** This is **not** true in general because, for example, it fails for $z_1 = i = z_2$. Indeed,

$$\operatorname{Im}(i \cdot i) = \operatorname{Im}(-1) = 0$$

whereas

$$\operatorname{Im}(i)\operatorname{Im}(i) = 1 \cdot 1 = 1.$$

Optional: In general,

$$\operatorname{Im}(z_1 z_2) = \operatorname{Im}\big(x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\big) = x_1y_2 + x_2y_1$$

whereas $(\operatorname{Im} z_1)(\operatorname{Im} z_2) = y_1y_2$. So the equality $\operatorname{Im}(z_1z_2) = (\operatorname{Im} z_1)(\operatorname{Im} z_2)$ will fail whenever $x_1y_2 + x_2y_1 \neq y_1y_2$. (Only **[7 pt]** if that's all you do.) It remains to find (at least) *one* particular example in which $x_1y_2 + x_2y_1 \neq y_1y_2$. The simplest is perhaps $z_1 = i = z_2$, which was used above.

5. [Page 19 identity (1-26)] **[20 pt]**

Method 1: Use basic properties. Namely, use: the definition $z_1/z_2 = z_1 z_2{}^{-1}$ together with the identities $z^{-1} = (1/|z|^2)\overline{z}$, $|zw| = |z|\,|w|$, and $|cw| = c|w|$ for a real $c > 0$. Then:

$$\left|\frac{z_1}{z_2}\right| = \left|z_1\left(\frac{1}{|z_2|^2}\overline{z_2}\right)\right| = \left|\frac{1}{|z_2|^2}(z_1\overline{z_2})\right| = \frac{1}{|z_2|^2}|z_1\overline{z_2}| = \frac{1}{|z_2|^2}|z_1|\,|\overline{z_2}| = \frac{1}{|z_2|^2}|z_1||z_2| = \frac{|z_1|}{|z_2|}$$

Method 2: Use polar form. (Only **[18 pt]** unless you include the "justification" mentioned below.)

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right| = \frac{r_1}{r_2}\left|e^{i\theta_1}e^{-i\theta_2}\right| = \frac{r_1}{r_2}\left|e^{i(\theta_1-\theta_2)}\right| = \frac{r_1}{r_2}\cdot 1 = \frac{|z_1|}{|z_2|}$$

Of course this requires justification, namely, that $e^{i\theta_1}/e^{i\theta_2} = e^{i(\theta_1-\theta_2)}$. But that's easy:

$$e^{i\theta_2}\,e^{i(\theta_1-\theta_2)} = e^{i\left(\theta_2+(\theta_1-\theta_2)\right)} = e^{i\theta_1}$$

Method 3: Use Cartesian coordinates and to hack out everything algebraically from scratch starting with $z_1 = x_1+iy_1, z_2 = x_2+iy_2$. This way is most unpleasant and does not reasonably exploit properties already established. (Only **[12 pt]** for this way.)