Math 421 • Fall 2010

The Factor Theorem and a corollary of the Fundamental Theorem of Algebra

27 August 2010

Prerequisites

Mathematica

Aside from having a working *Mathematica* system at your disposal and knowing how to type input, how to evaluate an Input cell, and how to navigate around a notebook, there are really no prerequisites.

In fact, working through this notebook is a good way to learn some *Mathematica* basics and even some more advanced *Mathematica* techniques.

All the input shown uses the display form ⅈ obtained by typing ESC`ii`ESC, but everything would work identically if you typed `I` instead.

As you work through this notebook in *Mathematica*, if you come across a *Mathematica* function you don't understand, try using the `?` information command to find out something about it. (And follow the ≫ hyperlink in it, if any, to Documentation Center help.) For example, to learn about `PolynomialQuotient`, used in the first section below, evaluate:

```
In[1]:= ?PolynomialQuotient
```

PolynomialQuotient[p, q, x] gives the quotient of
p and q, treated as polynomials in x, with any remainder dropped. ≫

David Park’s *Presentations* application is ***not*** needed for this notebook.

Mathematics

You need to know about adding and multiplying complex numbers.

Clearing a variable

Many *Mathematica* examples will use the variable `z`. Make sure no value has been assigned to `z` yet:

```
In[2]:= Clear[z]
```

The Division Theorem

You know from arithmetic that you can divide one positive integer a by another positive integer b to obtain an integer quotient q and a remainder integer r. That is, given positive integers a and b, there are unique integers q and r with

$$\frac{a}{b} = q + \frac{r}{b},$$

or, equivalently,

$$a = b\,q + r,$$

in each case with

$$0 \leq r < b.$$

the latter inequality says that the remainder r is less than the “divisor” b.

For example, if you use long division to divide 2356 by 14, you obtain a quotient of 168 and a remainder of 11, so that

$$2356 = 14 \times 168 + 11.$$

Mathematica: can verify that result for you:

```
In[3]:= 2356 == 14 × 168 + 3
Out[3]= False
```

And you can use *Mathematica* in the first place to calculate the quotient and remainder:

```
In[4]:= {Quotient[2356, 14], Mod[235, 14]}
Out[4]= {168, 11}
```

Likewise, you can use long division of polynomials to divide one polynomial $A(z)$ in a variable z by another polynomial $B(z)$ in that variable so as to obtain a quotient $Q(z)$ and a remainder $R(z)$, with the remainder having a smaller degree than that of the “divisor” $B(z)$. More precisely:

> **The Polynomial Division Theorem.** Let $A(z)$ and $B(z)$ polynomials in the variable z with real or complex coefficients and with $B(z)$ not the zero polynomial. Then there are unique polynomials $Q(z)$ and $R(z)$ in z with real or complex coefficients, respectively, such that
>
> $$A(z) = B(z)\,Q(z) + R(z)$$

and

$$\deg R(z) < \deg B(z).$$

For example,

$$\frac{z^4 - 4 z^3 + z + 6}{z^3 - 2} = (z - 4) + \frac{3 z - 2}{z^3 - 2},$$

or equivalently,

$$z^4 - 4 z^3 + z + 6 = \left(z^3 - 2\right)(z - 4) + (3 z - 2).$$

Again, *Mathematica* can verify that result…

```
In[5]:= z^4 - 4 z^3 + z + 6 == (z^3 - 2) (z - 4) + (3 z - 2) // Expand

Out[5]= True
```

…and *Mathematica* can calculate the polynomial quotient and remainder for you:

```
In[6]:= a[z_] := z^4 - 4 z^3 + z + 6;
        b[z] := z^3 - 2;
        {PolynomialQuotient[a[z], b[z], z], PolynomialRemainder[a[z], b[z], z]}

Out[8]= {-4 + z, -2 + 3 z}
```

Probably you've only used polynomial long division when both $A(z)$ and $B(z)$ have integer coefficients. In that case, the result still holds—and both the quotient $Q(z)$ and remainder $R(z)$ also have integer coefficients—provided that the coefficient of the highest power of z in the divisor $B(z)$ is 1.

A rigorous proof of the theorem, which is beyond the scope of this course, uses mathematical induction. The informal idea of the proof is what happens in long division of polynomials: at each step the degree of the remainder at that step has a lesser degree than does the remainder at the previous step, so that you can keep going until you reach a step where the degree of the remainder at that step is less than the degree of the divisor.

Exercise 1. With paper and pencil, carry out long division to express the given polynomial $A(z)$ in the form $B(z)\, Q(z) + R(z)$ for the given polynomial $B(z)$. Then use *Mathematica* to verify the result.

(a) $P(z) = z^3 - 5 z^2 - 4 z + 20$ and $Q(z) = z - 5$.

(b) $P(z) = z^3 - 5 z^2 - 4 z + 20$ and $Q(z) = z - 1$.

(b) $P(z) = z^5 - 12 z^4 + 48 z^3 - 62 z^2 - 33 z + 90$ and $Q(z) = z^2 - 3 z + 1$.

The Factor Theorem

The theorem is:

> **The Factor Theorem.** Let $P(z)$ be a polynomial in z (with real or complex coefficients) of degree $n > 0$. Then a (real or complex) number z_0 is a root of $P(z)$ if and only if
>
> $$P(z) = (z - z_0)\,Q(z)$$
>
> for some polynomial $Q(z)$ of degree $n - 1$.

Proof. First assume that z_0 is a root of $P(z)$. By long division, there is a quotient polynomial $Q(z)$ and a remainder polynomial $R(z)$ for which

$$P(z) = (z - z_0)\,Q(z) + R(z).$$

Since the divisor $z - z_0$ is of degree 1, the remainder polynomial $R(z)$ is of degree 0, that is, a constant r. Then

$$P(z) = (z - z_0)\,Q(z) + r.$$

Take $z = z_0$ in both sides of this equation to obtain

$$0 = P(z_0) = 0\,Q(z_0) + r,$$

so that $r = 0$. Hence

$$P(z) = (z - z_0)\,Q(z).$$

Since $P(z)$ has degree n whereas $z - z_0$ has degree 1, then $Q(z)$ has degree $n - 1$.

Conversely, assume that $P(z) = (z - z_0)\,Q(z)$ for a polynomial $Q(z)$ of degree $n - 1$. Take $z = z_0$ in this equation to obtain $P(z_0) = 0\,Q(z_0) = 0$. Hence z_0 is a root of $P(z)$. ∎

Example of the Factor Theorem

Make sure no value has been assigned to the variable `z` yet:

```
In[9]:= Clear[z]
```

Define a polynomial:

```
In[10]:= p[z_] := z^3 - 5 z^2 + 17 z - 13
```

By inspection, 1 is a root of this polynomial:

```
In[11]:= p[1]
Out[11]= 0
```

Give this root a name:

In[12]:=
```
z0 = 1;
```

The question is how to form the quotient polynomial $Q(z)$ in *Mathematica*. (As usual, begin user-defined *Mathematica* names with use lower-case letters.)

One way to obtain the quotient:

In[13]:=
```
q[z_] := PolynomialQuotient[p[z], z - z0, z]
q[z]
```

Out[14]= $13 - 4 z + z^2$

Another way to obtain the quotient:

In[15]:=
```
Simplify[p[z] / (z - z0)]
```

Out[15]= $13 - 4 z + z^2$

In turn find roots of the quotient:

In[16]:=
```
quadraticRoots = Solve[q[z] == 0, z]
```

Out[16]= {{z → 2 - 3 ⅈ}, {z → 2 + 3 ⅈ}}

Factor the quotient:

In[17]:=
```
Factor[q[z]]
```

Out[17]= $13 - 4 z + z^2$

Ugh! That does nothing. You have to tell *Mathematica* that you're looking for factoring "over the complex numbers". And you may do that as follows, using the `Extension` option to `Factor`:

In[18]:=
```
factoredQuotient = Factor[q[z], Extension → {ⅈ}]
```

Out[18]= ((-2 - 3 ⅈ) + z) ((-2 + 3 ⅈ) + z)

It so happens that the two complex roots of the quadratic quotient polynomial have integers as real and complex parts; in other words, these two complex roots belong to the set of **Gaussian integers**. Then you could also use `Factor` but with the option `GaussianIntegers → True`:

In[19]:=
```
Factor[q[z], GaussianIntegers → True]
```

Out[19]= ((-2 - 3 ⅈ) + z) ((-2 + 3 ⅈ) + z)

Factor the original cubic:

In[20]:=
```
factoredCubic = (z - z0) factoredQuotient
```

Out[20]= ((-2 - 3 ⅈ) + z) ((-2 + 3 ⅈ) + z) (-1 + z)

Exercise 2. Check that the product of three linear factors really is the original polynomial. Rather than looking back at the original cubic polynomial `p[z]`, you should do your checking by evaluating a suitable equation of the form

```
p[z]==…
```

and seeing that the result is `True`.

Exercise 3. Repeat the work that was done with the cubic polynomial $z^3 - 5z^2 + 17z - 13$ but now doing all the calculations with paper and pencil. (You'll need to carry out a "long division" of the cubic by the linear polynomial $z - 1$.)

Exercise 4. Use *Mathematica* to factor the original cubic polynomial all at once.

Exercise 5. With *Mathematica*, repeat for $z^3 - 3z^2 + z + 5$ what we did above for `p[z]`. Begin by observing that now -1 is a root.

Exercise 6. Repeat the preceding exercise but for $z^3 - i z^2 + z - i$. Begin by observing (and checking!) that i is a root.

So far, the example and exercises used only half of the Factor Theorem—that if z_0 is a root, then $z - z_0$ is a factor of the polynomial. But there is another half to the Factor Theorem, the converse: if $z - z_0$ is a factor of the polynomial, then z_0 is a root. The following exercise illustrates that half with an example.

Exercise 7. In *Mathematica:* Define the polynomial $p(z)$ to be $(z+2)\left(z^2 - 3z + 5\right)$. Multiply the linear factor and quadratic factors there to obtain an unfactored cubic polynomial. Finally, verify that -2 is a root of this unfactored cubic.

Alternate way to factor, given the roots (*advanced*)

There is another way to form the (unexpanded) product of factors of a polynomial if you already know its roots:

```
In[21]:= quadraticProduct = Apply[Times, z - (z /. quadraticRoots)]
Out[21]= ((-2 - 3 i) + z) ((-2 + 3 i) + z)
```

Let's analyze how that worked, step-by-step. Start with:

```
In[22]:= quadraticRoots
Out[22]= {{z → 2 - 3 ⅈ}, {z → 2 + 3 ⅈ}}
```

Use the replacement rules in the preceding output to obtain a list of the actual roots:

```
In[23]:= z /. quadraticRoots
Out[23]= {2 - 3 ⅈ, 2 + 3 ⅈ}
```

Form the corresponding linear polynomials:

```
In[24]:= z - (z /. quadraticRoots)
Out[24]= {(-2 + 3 ⅈ) + z, (-2 - 3 ⅈ) + z}
```

Finally, multiply those linear polynomials to obtain the factored quadratic:

```
In[25]:= Apply[Times, z - (z /. quadraticRoots)]
Out[25]= ((-2 - 3 ⅈ) + z) ((-2 + 3 ⅈ) + z)
```

That final step used the "functional programming" construct `Apply[Times,…]` in order to apply the `Times` (multiplication) function to the list of two linear polynomials.

Notice that *Mathematica* did *not* expand—multiply out—the product.

Corollary to the FTA

The following theorem is very important. Its proof is harder than you might at first suppose. One proof uses the theory of integrating a complex-valued function of a complex variable around a closed curve in the complex plane.

> **The Fundamental Theorem of Algebra** (**FTA**). Every non-constant polynomial with real or complex coefficients has at least one real or complex root.

Starting with that restatement of the FTA, a proof by mathematical induction establishes the following corollary.

> **Corollary.** Let $P(z)$ be a non-constant polynomial with complex coefficients, of degree n. Then $P(z)$ has exactly n (not necessarily distinct) complex roots.

The easiest way to make sense for now of the "not necessarily distinct" part of the conclusion there is to rephrase the corollary as follows:

Corollary (restated). Let $P(z)$ be a non-constant polynomial with complex coefficients, of degree n . Then there are n complex numbers $z_1, \ z_2, \ \ldots, \ z_n$, not necessarily different from one another, for which

$$P(z) = (z - z_0)(z - z_1) \cdots (z - z_n) \; .$$

The justification for that restatement is as follows. In view of the Factor Theorem and the FTA: If $p(z)$ is a non-constant polynomial, then there is a number z_0 and a polynomial $q(z)$ such that $p(z) = (z - z_0)\, q(z)$.

Example of the Corollary

Here is a polynomial of degree $n = 5$:

```
In[26]:= p[z_] := z^5 + (3 - 4 i) z^4 - (1 + 12 i) z^3 - (11 + 12 i) z^2 - (12 + 4 i) z - 4;
```

Factor it:

```
In[27]:= factoredQuintic = Factor[p[z]]
Out[27]= (-2 i + z)^2 (1 + z)^3
```

(Notice that the `Extension→{i}` option to `Factor` was not needed because the polynomial already had complex coefficients.)

Find the roots of the polynomial:

```
In[28]:= z /. Solve[p[z] == 0, z]
Out[28]= {-1, -1, -1, 2 i, 2 i}
```

There you see that the root -1 has "order 3", that is, appears three times in the list; similarly, the root $2\,i$ has "order 2". On the other hand, the linear polynomial $z - (-2)$ appears to order 3 in the factored polynomial; similarly, $z - (-2\,i)$ appears to order 2. This situation is precisely as predicted by the Corollary.

Exercise 8. Carry out a similar confirmation of the Corollary with each of the following polynomials:

(a) $z^6 - 6\,z^5 + 11\,z^4 + 4\,z^3 - 29\,z^2 + 10\,z + 25$.

(b) $z^5 - z^4 - 9\,z^3 + 5\,z^2 + 16\,z - 12$.

(c) $z^5 - (4 + i)\,z^4 + (7 + 2\,i)\,z^3 - (8 + i)\,z^2 + 6\,z - 2$.