Developmental Review 67 (2023) 101062

Contents lists available at ScienceDirect

Developmental Review

journal homepage: www.elsevier.com/locate/dr

# Arithmetic thinking as the basis of children's generative number concepts

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ARTICLE INFO

*Keywords:*
Natural number acquisition
Number concepts
Numeration systems
Recursion
Successor function
Theories of mathematical cognition

ABSTRACT

Predominant psychological theories of number acquisition posit that children acquire natural number concepts as they acquire the successor principle, or the knowledge that every natural number is succeeded by another natural number that is exactly-one more than it. However, exactly how children acquire the successor principle remains largely unexplained. Recently developed ideas within this family of theories posit that an abstract recursive successor function is acquired from the recursive structure of number words; however, the types of recursion underlying the successor function and number words are distinctively different (one is a self-referential function and the other is a self-embedded structure), making it difficult to theorize how one type triggers the acquisition of another. Moreover, our analysis of the literature questions if the knowledge about the successor principle is even empirically measurable. Here, we argue that number acquisition is a process of understanding a generative rule that governs the system of natural numbers and point out that the successor principle is not the only generative rule that governs the natural number system. We propose an alternative hypothesis that generative number concepts emerge from children's realization about how the combinatorial rules of numerals allow arithmetic (specifically additive and multiplicative) representations of quantity. Importantly, under addition and multiplication—which are historically rooted in concatenation and grouping of physical objects—natural numbers are mathematically closed. As a corollary, the system of infinitely generative natural numbers is conceptualized. This new theoretical framework allows the construction of novel empirical questions and testable hypotheses based on the formalized rules of numerical syntax and numeration systems, and therefore opens a new avenue for studying later stages of children's acquisition of number concepts.

## Introduction

Natural numbers make up a quintessential symbolic system that characterizes the uniqueness of human cognition (Deacon, 1999; Wiese, 2003). They are built based on generative rules that define their relations and represent potentially infinite discrete quantities. Natural number concepts are simple and straightforward, or easy as, "1, 2, 3." Despite this simplicity, it takes a child many years to master number concepts (Fuson, 1988a; Geary, 2006; Sophian, 2017; Steffe et al., 1983), and without appropriate cultural and symbolic resources, one may never learn them (Everett & Madora, 2012; Flaherty & Senghas, 2011; Frank et al., 2008; Gordon, 2004;

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https://doi.org/10.1016/j.dr.2022.101062
Received 15 March 2022; Received in revised form 22 November 2022;
Available online 6 December 2022

Pica et al., 2004; Pitt et al., 2022; Schlaudt, 2020). Acquiring number concepts is a major conceptual breakthrough that every child in modern society experiences, yet no other species can acquire these seemingly simple ideas. How the human mind achieves this acquisition has been a central cognitive and developmental science question for a long time.

The conceptual breakthrough in number acquisition starts with counting, and decades of empirical work has shown that children demonstrate interesting developmental patterns during early childhood. For example, children undergo various levels of knowledge in which they can understand up to and only up to certain numbers, and then at some point they learn that the last number word used to count a set refers to the numerical value of the set (Carey, 2009; Condry & Spelke, 2008; Fuson, 1988a; Geary et al., 2019; Gelman & Gallistel, 1978; Gunderson et al., 2015; Le Corre & Carey, 2007; Wynn, 1990, 1992). Such descriptive developmental patterns of number acquisition have attracted many researchers to study the cognitive and developmental mechanisms for understanding the meaning of number words; however, most research in this field has focused on children's acquisition of small numbers, typically from one to ten in most languages with modern numeration systems. Relative to small number acquisition, little is known and theorized about later stages of number acquisition. We are interested in these later stages of development, specifically when children acquire the knowledge that a set of rules allows the generation of new numbers indefinitely.

In this paper, we develop a novel theoretical proposal concerning children's acquisition of *generative number concepts*. We first review previous psychological theories of number acquisition and analyze how those theories explain the acquisition of generative number concepts. As will be seen, most theories explain it as a process of acquiring the *successor principle*, according to which any nature number, $N$, has its successor, $N + 1$, that is also a natural number. We develop a criticism of the successor-principle-based theories and propose an alternative approach to theorizing generative number concepts. Namely, we propose that generative number concepts emerge from children's realization about how the combinatorial rules of numerals allow arithmetic (specifically additive and multiplicative) representations of quantity.

## Natural number concepts

Before we begin, we provide an operational definition of natural number concepts (i.e., knowledge obtained at the end the acquisition process that most adults have) as the mental representation of the cardinal value of a discrete set (von Mengden, 2010; Wiese, 2003). Accordingly, each number—as humans conceptualize—is distinct from one another and is uniquely positioned in a well-ordered sequence (Gelman & Gallistel, 1978). As will be argued below, sets of numerals that make up an ordered sequence of well-distinguished elements are indispensable to natural number concepts. The relative position of each number within the entire system of numbers is determined by set(s) of rules, which allow the numbers to be generative. We use the term *generative* number concepts to describe the understanding about this rule-governed system of natural numbers that can be generated indefinitely.

Next, we also clarify some of the terminologies that are frequently used in this paper. First, natural numbers (henceforth numbers for simplicity) represent a class of entities which are assigned to describe the size of discrete sets, whereas numerals are written or verbal symbols that human populations use to signify numbers (e.g., "1", "one"). The most frequently used systems of numerals, or numeration systems, in modern society are systems based on number words and Arabic numerals. Number words can be divided into cardinal number words, which refer to the quantity (e.g., one, two, three), and ordinal number words, which refer to order (e.g., first, second, third). This work focuses on cardinal number words. Cardinal number words can be further divided into simplex and complex cardinal number words. Simplex cardinal number words, hereafter called simplex number words, refer to number words made up of one morpheme (e.g., one, ten). Complex cardinal number words, hereafter called complex number words, refer to number words that are morpho-syntactically composed of two or more simplex number words (e.g., seventy, two hundred). There exist non-systemic cardinal specific quantifiers such as 'dozen' or other subcategories of number words such as 'fourfold,' although they remain outside of this work. Finally, we use the term external representation to refer to numerals or supplementary quantities (physical objects that help enumeration such as tallies and pebbles) that reside outside of the mind and signify number. In contrast, the term internal representation refers to mental objects or models with semantic properties—in our context, cardinal value.

## Existing psychological theories of number acquisition

As mentioned, the aim of this paper is to develop a theoretical proposal concerning children's acquisition of generative number concepts. To do so, we first review previous major psychological theories of number acquisition, with a focus on how they explain the acquisition of generative number concepts.

### *One-to-one correspondence as a foundational principle for number concepts*

Humans have long used supplementary quantities (such as tallies and pebbles) to enumerate, as evidenced by archeological and anthropological artifacts dating back tens of thousands of years (d'Errico et al., 2018; Dantzig, 1954; Ifrah, 2000; Menninger, 1969; Schlaudt, 2020). In fact, even the words we use to talk about numbers and counting have origins in supplementary quantities. For example, the words tally and calculate originate from the Latin words for cutting (*talea)* and pebble (*calculus*) respectively. These physical objects were used as external representations of quantity and facilitated the comparison of set quantities using the mental operation of one-to-one correspondence (i.e., matching sets), giving rise to the concept of equinumerosity.

The acquisition of one-to-one correspondence has been a target of investigation since Piaget's studies about the conservation of number (Piaget, 1965). Gelman and Gallistel (1978) theorized this to be one of the key principles of counting or "rules of procedure" that "involves the ticking off of the items in an array with distinct ticks (tags, numerons, numerlogs) in such a way that one and only one tick is used for each item in the array" (p. 77). That is, any correct counting procedure requires that each element in the set of counted objects is mapped once and only once with an external or internal label representing a quantity. Even when counting with labels is not considered, equinumerosity established by this one-to-one correspondence has been theorized to serve as a developmental foundation for number concepts. For example, Izard and colleagues (Izard et al., 2008, 2014; Pica et al., 2004) proposed that children's comprehension of one-to-one correspondence between sets requires several principles. These principles include understanding: that equinumerosity between two sets is not affected by spatial reorganization of items within any set (i.e., identity principle); that if two sets are equal in number then the addition or subtraction of one element in a set disrupts their equinumerosity (i.e., addition/subtraction principle); and that the substitution of one element in a set does not disrupt equinumerosity (i.e., substitution principle). Importantly, the acquisition of these principles does not require external symbols, counting, nor an ability to approximately estimate large numbers. For instance, three-year-old children, who do not yet understand exact numbers beyond four, can nevertheless reconstruct exact quantities in a set of 5 or 6 objects using one-to-one correspondence as long as the identity of the objects remain unaffected (Izard et al., 2014).

Numerical equality derived from one-to-one correspondence, however, is not sufficient for establishing number concepts, as defined earlier. This is because one-to-one correspondence relies on external representations. More precisely, one-to-one correspondence does not allow the internal representation of cardinal values beyond the memory or attention capacity without referring to supplementary quantities (Menninger, 1969). Such an insufficiency is observed in many empirical studies and ethnographic records. Young children who do not understand number words beyond "four" fail to understand numerical equality between two sets when the identity of the elements in a set changes (Izard et al., 2014). Even children who understand number words well beyond "four" tend to fail at matching sets of more than 5 items when counting is prevented (Schneider et al., 2022). Adult Deaf signers without a conventional lexicon for numbers are not able to communicate or express exact numerical quantities greater than three (Spaepen et al., 2011, 2013). Indigenous adults' representation of exact cardinal values are limited to the number words that they can recite (Pitt et al., 2022). This limitation is even true for the Oksapmin numeration system, which represents cardinalities up to 27 using one-to-one correspondence with body parts in a stable sequence (Bender & Beller, 2011; Saxe, 1981). Moreover, even literate adults in a modern society find it difficult to produce exact numerical quantities based on one-to-one correspondence when counting is made unavailable by a concurrent verbal shadowing task (Frank et al., 2012).

It is important to note that one-to-one correspondence in this context is different from the logico-philosophical definition of one-to-one correspondence, introduced as Hume's principle (Decock, 2008). That is, the limitation of one-to-one correspondence in building the concept of natural numbers is not a logical argument but rather a cognitive argument. Without distinct labels (e.g., an ordered count list) to make a one-to-one correspondence between the elements of the external set and internal representations, the constraints in memory and attention limit human's capacity to judge simultaneously that for every element in a set, there is exactly-one element in the other set. In sum, while one-to-one correspondence may be critical for the comprehension of exact equality in small numbers, evidence suggests that neither adults using a restricted numeration system nor children with limited counting abilities develop generative number concepts based solely on one-to-one correspondence.

### *Natural number knowledge from the analog magnitude system*

One of the earliest theories of natural number acquisition assumes that number knowledge is an intrinsic property of the animal mind based on the analog magnitude system, an evolutionarily ancient system for representing magnitudes approximately (Brannon, 2005; Dehaene, 2001; Gallistel & Gelman, 2005; Leslie et al., 2008; Piazza et al., 2007). In the original formulation of the analog magnitude system, nonhuman animals are thought to represent external quantities as a continuous internal magnitude using a mental accumulator (Meck & Church, 1983), which gained some empirical support in the studies of nonhuman primates (Brannon & Terrace, 1998, 2000; Smith et al., 2003). However, as also shown in earlier studies in preverbal infants and young children (Loosbroek & Smitsman, 1990; Starkey et al., 1990; Starkey & Cooper, 1980), this nonverbal or preverbal representation of the mental accumulator is assumed to be approximate as the internal magnitude for each item is non-precise (Gallistel & Gelman, 1992). It is worth noting that some recent theoretical developments challenge this interpretation and argue for the preciseness of the analog magnitude system (Clarke & Beck, 2021; Halberda, 2016), although it should be emphasized that the analog magnitude system alone, as empirical data from nonhuman animals demonstrate, is not sufficient for the emergence of the natural number knowledge.

Gallistel and Gelman (1992) posited that children's acquisition of number knowledge is a process of recalibration of those preverbal representation of number. Specifically, the authors reasoned that counting with cardinal number words is structurally equivalent to the mechanics of the mental accumulator and that this isomorphism between the two systems drives the acquisition of the numerical meaning of the number words. In sum, this theory states that learning natural numbers is a reorganization of existing knowledge—analog magnitude representation—rather than the acquisition of a new representational system (Rips et al., 2013).

This theory has been challenged and criticized from many angles. Most critically, it does not explain how the analog representation is compatible with the representation and computation of discrete magnitude (Carey & Barner, 2019). Not surprisingly, there is no empirical evidence to support the idea that the analog magnitude system directly enables the acquisition of precise number knowledge, and even indirect evidence for correlations between children's representation of analog magnitude and their symbolic number knowledge remains very weak (Carey & Barner, 2019; Chen & Li, 2014; Fazio et al., 2014; Leibovich & Ansari, 2016; Sokolowski et al., 2017; Wilkey & Ansari, 2020).

*Recursive successor function as an innate mechanism for generating integers*

As an important extension or revision to the original model proposed by Gallistel and Gelman (1992), Leslie et al. (2008) proposed that an innate mechanism called the "integer symbol generator" that generates mental symbols to represent discrete quantities. According to this proposal, this mechanism is enabled by two elements: the primitive representation of the cardinal value 1 and the successor function that computes the operation $N + 1$. These properties are equivalent to Giuseppe Peano's original formulation of the arithmetic properties of natural numbers (Peano, 1977), referred to as Peano or Dedekind-Peano axioms.

The first of Peano axioms (1) states that the constant 1 is an element in the set of natural numbers (*N*). In modern formulations, the number 0 is instead used as the constant in the first axiom, but we assume the constant 1 in this work because empirically speaking number acquisition begins from counting that starts from 1 rather than 0. The second axiom (2) introduces the successor function (*S*) and asserts that for all elements (*x*) in N, there exists an element *y* whose predecessor is *x* (originally the 6th axiom in Peano, 1977). Then, the axiom of induction (3) states that for any property F that holds for the first element (F is true for the number 1) and for a number n, then F also holds for $n + 1$, and therefore F will apply to all natural numbers. A consequence of this last axiom is that any natural number can be obtained by applying the successor function recursively to the number 1, and as a corollary, the set *N* is infinite.[1]

(1) $\mathrm{N}1$
(2) $\forall x(\mathrm{N}x \rightarrow \exists y\ x\mathrm{P}y)$
(3) $\forall \mathrm{F}[\mathrm{F}1 \wedge \forall xy(\mathrm{F}x \wedge x\mathrm{S}y \rightarrow \mathrm{F}y) \rightarrow \forall x(\mathrm{N}x \rightarrow \mathrm{F}x)]$

The model proposed by Leslie et al. (2008) is structurally equivalent to the Peano axioms. First, the primitive 1 posited in Leslie and colleagues' model is equivalent to the first of Peano axioms (1). Second, the integer generator in Leslie and colleagues' model is a mechanism that mimics the successor function in Peano axioms (2). The authors propose that the integer generator allows the brain to generate an "internal compact code" for representing integer values to effectively realize an unbounded set of numbers. Importantly, both the successor function and the integer generator follow the logical-mathematical definition of a recursive function, whose principle property is self-reference (Lobina, 2019; Soare, 1996; Tomalin, 2006), a point which we will return to later.

While the idea of the innate integer generator has a strong theoretical background, as it resembles how natural numbers are formally defined, there is no empirical evidence, at least to our knowledge, that directly supports the existence of this mechanism. Instead, cross-linguistic and developmental studies provide evidence against such an idea. First, while most human populations represent the cardinal value of 1 using a distinct numeral (Corbett, 2000; Da Silva-Sinha et al., 2017; Hurford, 1975; Leko, 2009; Martínez, 1999), there exists indigenous people who do not represent the cardinal value of 1 precisely. In the Mundurucu numeration system, the cardinal number word one (*Pug*) represents approximately one (Pica & Lecomte, 2008). Second, many indigenous human populations using isolated languages use cardinal number words only to represent small exact cardinalities without developing a full numerical system (Comrie, 2013; Corbett, 2000; Ifrah, 2000). For instance, in Aikanã, the simplex number words for 1 (*Amẽmẽ*) and 2 (*Atuca*) make up the words for 3 (*Atuca amẽmẽ*), 4 (*Atuca atuca*), and 5 (*Atuca atuca amẽmẽ*) (Da Silva-Sinha et al. 2017). We speculate that these number words are typically not used in a sequence because the additive nature of these number words, built from reduplication, could generate confusion (Is the sequence *Amẽmẽ Atuca Atuca Amẽmẽ* interpreted as 1 2 3 or 1 4 1?). Third, in developmental studies, children who are capable of using the rote counting procedure nevertheless fail to name the next cardinal value in a paradigm that does not involve rote counting (Schneider et al., 2021; Spaepen et al., 2018). These findings together challenge the proposition of an innate representation of the number 1 and an innate recursive mechanism for representing the next cardinal value.

At this point, it is important to be reminded of the distinction between number and numeral, expressed as compact integer code and compact notational system, respectively, in Leslie et al. (2008). Although Leslie and colleagues do not provide a detailed proposal for how the two are related, one may still argue that the compact integer code exists in all humans but that the reason some cultures do not have a representation of the number 1 or 6 is because they do not have the correct numeral for those symbols. In other words, one may argue that lacking *external* integer representations is not a proof against having *internal* integer representations. Nevertheless, it is difficult to reason why those numerals would not exist if the compact integer code does exist. Let's assume that, following Leslie and colleagues, the compact integer code represents the numbers 6 even in Aikanã because these internal representations are independent of the numeral. However, notice that the increment from 2 to 3 and from 4 to 5 is marked by a concatenation of the numeral for 1 (i.e., *Amẽmẽ*), making the $N + 1$ operation explicit. Yet, this rule does not seem to be sufficient for constructing the successor of the number 5, like atuca atuca atuca (two two two), which would be expected if the integer generator existed. What prevents this language from doing so? Of course, one may argue that people in such cultures never felt the need to devise a numeral to represent numbers beyond some point. This means, however, that the innate mechanism for representing integers (if it exists) is not sufficient for generative number concepts. Rather, it means that cultural pressure is necessary for number concepts and much of academic mathematics (e.g., Geary, 1995). Another problem of this line of reasoning is that it makes the model largely unfalsifiable. In fact, it seems difficult, if not impossible, to study internal number concepts empirically without hinging on the numerals (as also discussed in our operational definition of natural number concepts), especially during the acquisition phase in children. That said, we argue that a psychological theory of number acquisition must provide a sufficiently detailed explanation for *how* the knowledge about numerals arise from the concept of numbers or vice versa (a similar critique is well elaborated in Carey, 2009).

[1] The logic symbol ∀ represents universal quantification and is read as "for all." ∃ represents existential quantification which is read as "there exists." ∧ represents logical conjunction. Finally, → is a conditional connective for two propositions and is read as "if … then.".

*Recursive successor function as a product of inductive inference*

Perhaps the most dominant theory of number acquisition was originally developed by Carey and colleagues based on the idea of Quinian boostrapping (Carey, 2004). Two main ingredients are critical for this theory. The first is an attentional system called the object file or parallel individuation system (Carey & Xu, 2001), by which preverbal infants and children distinguish up to three items contained in small sets (Izard et al., 2009; Lipton & Spelke, 2004; Loosbroek & Smitsman, 1990; Starkey et al., 1990; Starkey & Cooper, 1980; Xu et al., 2005). It is now widely acknowledged that humans, and nonhuman primates alike, have the capacity to simultaneously process up to, when randomly arranged, three or four items in a set (named subitizing) both visually and auditorily (Anobile et al., 2012, 2016; Burr et al., 2010; Piazza et al., 2011). The second critical component to this theory is an ordered sequence of count words. In most economically developed cultures, children experience and learn to recite the count list (usually from 1 to 10) at a very young age before they understand the cardinal value of the number words (Fuson, 1988b; Le Corre & Carey, 2007; Leybaert & Van Cutsem, 2002; Marušič et al., 2016; Wynn, 1990, 1992).

According to this theory, when a child attends to one individual object in their perceptual space and simultaneously listens to the utterance *one*, the parallel individuation system builds an arbitrary one-member set in working memory. A child then learns that *one* is the verbal label which represents all the sets with exactly one object. Acquiring the meaning of *two* and *three* is facilitated by a similar process; the number word utterance is associated with a corresponding set in working memory. Finally, through comparing the cardinalities of *one, two*, and *three* in the count list, they infer that as the representation of a set in working memory increases by one, the label in the count list advances by one. This observation results in the inference that the same operation applies to the rest of the labels in the count list. Thus, the initially meaningless number words serve as placeholders and, as it is argued, eventually represent cardinal values via semantic induction. This process is characterized as Quinian bootstrapping, according to which representational resources that transcend their input are constructed (see also Gentner, 2010).

Evidence for this theory mostly comes from studies that implement the Give-a-number task (Give-N), in which children are asked to give a specific number of items to the experimenter (Fuson, 1988a; Gelman & Gallistel, 1978; Greeno et al., 1984; Schaeffer et al., 1974; Wynn, 1990, 1992). Young children go through years of "subset-knower" stages during which their understanding of the meaning of number words is limited (Le Corre et al., 2006; Le Corre & Carey, 2007; Marušič et al., 2016; Wagner et al., 2015; Wynn, 1990), as if they are accumulating evidence needed to make inductive inference during this time. After children acquire the meaning of *four*, they typically understand the cardinality principle (CP), or that the last number word used while counting a set indicates the asked number of items (Gelman & Gallistel, 1978; Le Corre & Carey, 2007). At this point, the learning of the number words beyond *four* and up to the child's count list (typically tested up to *seven* or *eight*) happens instantaneously (but see Rousselle & Vossius, 2021). This pattern is consistent with the idea that children make an inductive inference based on the first few number words.

However, there is some lack of clarity in the formulation of this theory. Specifically, this theory uses the term successor function to explain the acquisition mechanism, but the term conflates at least two different ideas. First, successor function is used to refer to the rule governing the system of natural numbers, as in Leslie et al. (2008). This is apparent from the explicit reference to Peano axioms (Carey & Barner, 2019) and statements like "we must have the capacity to implement the successor function somehow if we are to represent natural number" (p. 222, Carey, 2009). However, in other parts of the theory, the term successor function is used interchangeably with the term inductive inference. Inductive inference in this context describes the process by which children understand the cardinality principle from the successive relations of objects (physical or mental) in a set and an ordered sequence of count words. Nevertheless, the result of inductive inference does not refer to the understanding about the system of natural numbers. Carey (2009) admits as much, "When children have learned to use a count list that extends to 10 or 20 to enumerate sets, they … may not have yet abstracted a concept number at all" (p. 252). In sum, successor function is, on the one hand, referred to the abstract system of natural numbers but, on the other hand, referred to the process (i.e., inductive inference) of acquiring CP knowledge. Yet, acquiring CP is not considered to have acquired the concept of natural numbers. Thus, at least one of the two ways in which the term successor is used must be inappropriate.

In fact, studies suggest that the acquisition of CP knowledge, which was originally considered to indicate the successor principle, does not seem to suggest so anymore. Lately, children's understanding of the "successor function" has been tested using the Unit task, a task which resembles how semantic induction works (Davidson et al., 2012; Sarnecka & Carey, 2008). In this task, children are simultaneously shown and told about $N$ items in an opaque container. Then, the experimenter adds one more item into the container, after which children are asked whether there are $N + 1$ or $N + 2$ items in the container. Previous studies have shown that young children with little knowledge about the counting sequence are able to update the mental representation of a small set being added or subtracted by items, indicating an early capacity for quantitative reasoning (Starkey, 1992). The Unit task, however, requires an explicit linkage between such quantitative reasoning and the count sequence. If children understand that advancing the cardinal value by one (i.e., adding one item to the container) is equivalent to moving one label in the count list (i.e., mapping between item relations and count list), they should be able to perform this task reliably. In the original study (Sarnecka & Carey, 2008), only CP knowers, but not subset knowers, reliably solved the Unit task with $N$ of 4 and 5 (i.e., asking for 4 + 1, 4 + 2, 5 + 1, and 5 + 2). This pattern was consistent with the idea that CP knowledge is acquired by mastering inductive inference which therefore makes them capable of representing the next number.

However, subsequent studies show conflicting results (Cheung et al., 2017; Davidson et al., 2012; Schneider et al., 2021; Spaepen et al., 2018). In Davidson et al. (2012), 84 CP knowers (aged 3;4 to 5;3) were asked to count as high as possible and were tested with the Unit task but this time with greater values of $N$ (using 4, 5, 13, 14, 15, 24, and 25). Unlike the original findings by Sarnecka & Carey (2008), many of those CP knowers failed the Unit task, even when the probed $N$ was within their highest count. For example, the majority of those who counted up to the 10–19 range performed at chance when asked about 4 + 1 or 5 + 1. Likewise, the majority of those who counted up to the 20–29 range performed at chance when asked about 13 + 1, 14 + 1, or 15 + 1. Similar results were shown in Spaepen et al. (2018).

These findings suggest that semantic induction is not responsible for the transition to having CP knowledge nor is semantic induction is a one-shot process. Rather, it takes time and practice to infer the cardinality relations across broader ranges of numerical magnitudes. Thus, using the term induction may not even be appropriate to describe these processes, although we keep this term in the remainder of this paper to be consistent with the literature (see also footnote 2). These recent findings are not surprising because semantic induction is about realizing the relations between mental representation of object and the count list. Therefore, the inferential process must be constrained by the limits of the mental representation of a set or the count list. For instance, if one's count list ends or becomes fuzzy at some number for whatever reason, that would constrain a possible inferential process (e.g., see Pitt et al., 2022). Likewise, if one cannot form a mental representation of objects or have sources of interference for forming it yet is asked to imagine them (as in the probing phase of the Unit task), this inferential process may easily break down. In sum, the collective evidence in the literature suggests that children go through multiple phases of realizing this inductive inference, not just at the time of CP transition, during their number acquisition path.

The remaining question, then, is how such a piecemeal acquisition of inductive inference gives rise to natural number concepts. Although an explanation for this question is not explicitly developed in the original formulation of Carey's theory (2009), reference to "a fully recursive successor function that generates an infinite set of numbers" (Carey & Barner, 2019) implies that repeated realization of inductive inference somehow triggers the successor principle in the child's mind.

### *Recursion in cardinal number words triggers recursion in natural numbers*

Recently, Barner (2017) has made an explicit proposal about how inductive inference implemented by counting triggers the successor principle. In this proposal, recursive regularities in number words, especially what is called the "productive decade + unit rule," trigger the recursive successor function in the child's mind (see also Schneider et al., 2021). According to this idea, children learn that semantic induction over a local sequence of number words[2] (what is called "item-based successor mapping") can be made repeatedly in many decades. From this induction process, children acquire "generalized successor knowledge" (Schneider et al., 2021), which we assume refers to the successor principle.

This idea has often been tested in studies using a rote counting procedure (Cheung et al., 2017; Chu et al., 2020; Davidson et al., 2012; Schneider et al., 2021). It has long been known that, when children are asked to rote count, they tend to stop counting at the end of a decade (e.g., thirty-nine or forty-nine) rather than in the middle of a decade, and that it takes years for children to master the regularities of the count list (Gervasoni, 2003; Gould, 2017; Guerrero et al., 2020; Miller & Stigler, 1987; Siegler & Robinson, 1982; Song & Ginsburg, 1988). Recent studies have shown an association between children's understanding of these regularities and other types of numerical knowledge. For example, children who are able to count up closer to one hundred demonstrate explicit knowledge about the endlessness of number, which was assessed by asking about the biggest number they knew and then about the result of adding 1 to it (Cheung et al., 2017). Similarly, children characterized as "productive counters"—who could count up to one hundred when given extra help to overcome decade boundaries—show better performance in other number tasks (Chu et al., 2020). Based on these findings, the authors claim that knowledge about the regularities in cardinal number words, more specifically knowledge of the decade + unit rule, leads to the acquisition of the successor principle.

This logic may be appealing as the basic argument is that recursion in one form triggers recursion in another form; however, an evaluation of this claim requires a precise understanding of the definition of recursion in these contexts.

## Challenges to number acquisition theories rooted in the successor principle

As we reviewed so far, most psychological theories concerning generative number concepts assume that one culminating point of children's acquisition of natural number concepts is the acquisition of the successor principle. Our critical review has revealed theoretical and empirical challenges for those successor principle-based theories. In this section, we further elaborate these challenges, which sets up the stage for the development of an alternative proposal in the next section, where we will propose that combinatorial rules of the numerals which allow additive and multiplicative thinking enable the acquisition of the generative rules of natural numbers.

### *Is recursion in complex number words equivalent to recursion in natural numbers?*

We first point out that, while recursion may sound like a unitary idea, theoretical analyses demonstrate that it is not (Hauser et al.,

[2] We acknowledge that the use of the term induction to describe this inferential process within a local sequence of number words may be inappropriate. This is because the term induction gives the interpretation that the observed pattern or rule *generalizes* indefinitely, but that is certainly not how we and others are using the term. Second, the term induction implies that we understand the *mechanism* (that is, beyond the phenomenon) underlying children's enumeration behavior or response in a number task, and we believe this is a stronger claim than what is permitted by the current literature. Alternatively, the term analogical mapping (Gentner, 2010; Holyoak & Thagard, 1989) may be used to refer to the phenomenon, making a more theory-neutral interpretation. However, for now we chose to keep the terms semantic induction or inductive inference because the idea of *generalization* albeit within a local sequence may be best explained as an inductive process (Carey, 2004; 2009).

2010; Lobina, 2011, 2014, 2019; Luuk & Luuk, 2011; Martins, 2012; Tomalin, 2006, 2011; Watumull et al., 2014). There are at least two distinct ways of defining recursion: by self-reference and by self-embeddedness. In this section, we describe how recursion in complex number words (i.e., number words that are morpho-syntactically composed of two or more simplex number words) is not equivalent to recursion in natural numbers, which debilitates the idea that the former triggers the latter (Barner, 2017).

From a logico-mathematical perspective, the main property of recursion is *self-reference* (Lobina, 2014, 2019; Soare, 1996; Tomalin, 2006). Self-reference is characterized as a recursive function, or mental operation that associates an input to some output. It requires two elements: values calculated in a previous step and the self-calling of the function itself. When applied to the number domain, it becomes immediately apparent that the successor function implicated in Peano axioms (2) follows this definition of recursion. For instance, applying the successor function, $S$, to the outcome of the function twice, starting from the initial value of 1, results in the number 3 (4).

(4) $3 = S(2) = S(S(1))$

In linguistics literature, recursion is often characterized as the product of binary Merge operation (Berwick & Chomsky, 2016; Chomsky, 2014, 2015) and Labeling algorithm (Cecchetto & Donati, 2015). Merge is a self-referential function that builds hierarchically structured expressions using syntactic objects. Merge takes the syntactics objects α and β and forms the set {α, β}. The output δ (i.e., {α, β}) is further capable of merging with a new syntactic object γ to form the set {γ, δ} (i.e., {γ, {α, β}}). Labeling is the algorithm that assigns a label to syntactic objects. Recursion in this context occurs when a syntactic object of one label is embedded in a syntactic object with the same label either directly or via another syntactic object with a different label. This definition considers recursion[3] as the property of a structure characterized by self-embeddedness, in contrast with the self-referential function defined in logico-mathematics as described above (Coolidge et al., 2011; Hollebrandse & Roeper, 2014; Pérez-Leroux et al., 2012; Pinker & Jackendoff, 2005; Roeper & Snyder, 2005; Terunuma et al., 2017; Terunuma & Nakato, 2018). In other words, self-embeddedness characterizes mental representation which can be distinguished from mental operation.

Seminal works by Hurford (1975, 1987, 2007) on the linguistic analysis of numerals illustrates recursive self-embedding. According to Hurford (1975), all developed numeration systems share a set of phrase structure rules (5).

(5) a. Number → Digit
 b. Number → Phrase (Number)
 c. Phrase → (Number) Multiplier

According to these rules, a Number can be constructed from a Digit (5a) which refers to lexical primitives including *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, and *nine* in English. Hence, the meaning associated with this local sequence of number words is essential for the representation of a numeral. Importantly, a Number can also be constructed from a Phrase optionally followed by another Number (5b), while a Phrase consists of Multiplier optionally preceded by a Number (5c). These latter rules construct a hierarchical structure of complex number words (see Fig. 1) with the syntactic category of Number embedded inside another Number either directly (5b) or via another category Phrase (5c). As will be discussed in a later section, the former rule allows an additive interpretation between a Phrase and a Number (e.g., sixty + two) and the latter rule allows a multiplicative interpretation between Number and Multiplier (e.g., two × hundred). Hereafter, we refer to these linguistic structural rules of numerals as numerical syntax.

As one way to evaluate the proposition developed in Barner (2017) (that recursion in complex number words triggers the recursive successor principle), we ask if recursion in numerals of modern numeration systems is equivalent to recursion in natural numbers. Before we start, it should be noted that the "decade + unit" rule is described as the units (i.e., labels from one to nine or lexical primitives) recursively repeated in the same order across every decade (Schneider et al., 2021). Unlike Hurford's (1976) analyses of numerals, this description lacks formalism and makes it unclear exactly what aspect of that rule is recursive or even if the rule could be considered recursive. Nevertheless, we proceed our analysis with the assumption that the recursive structure in the count list postulated by Barner (2017) is characterized as self-embedded recursion akin to that in Hurford's system.

The proposal that children acquire the recursive successor principle "when they discover the structure of the count list and that numbers are generated by a recursive base system" (p. 28) (Barner, 2017) conflates two different conceptualizations of recursion. One way recursion is described, which is used when explaining the successor principle, aligns with the definition of recursion in the logico-mathematical sense (Lobina, 2011; Soare, 1996). According to this definition, recursion is the property of a function not a structure. The other way recursion is described, which is used when explaining the structure of the count list, aligns with the definition of recursion which specifies a hierarchical organization of self-embedded syntactic structures (Lobina, 2011; Luuk & Luuk, 2011). Hence, the proposal that the recursive structure of complex number words triggers the acquisition of the recursive successor function implicitly assumes that self-embedded numeral structures trigger the acquisition of an abstract self-referential function. This implicit assumption is highly questionable. For example, the self-embedded structure of the numeral $_{\text{Number}}[\ _{\text{Phrase}}[\ _{\text{Number}}[\ _{\text{Digit}}$ *two* ] $_{\text{Multiplier}}$ *hundred* ]] better corresponds with the multiplicative operation 2 × 100 than with the outcome of the self-referential successor function $200 = S(199) = S(S(198)) = \ldots = S(\ldots S(1))$. Likewise, the structure of $_{\text{Number}}[\ _{\text{Phrase}}[$*two hundred* ] $_{\text{Number}}[\ _{\text{Digit}}$ *five* ]] better

[3] Some authors state that this is not a correct use of the term recursion. They argue that recursion is an induction schema defined as a function, but in this context recursion is used as the property of a data structure (Lobina, 2011, 2014, 2019; Luuk & Luuk, 2011; Tomalin, 2011; Watumull et al., 2014). In this paper, however, we follow the linguistic convention and call this recursion.

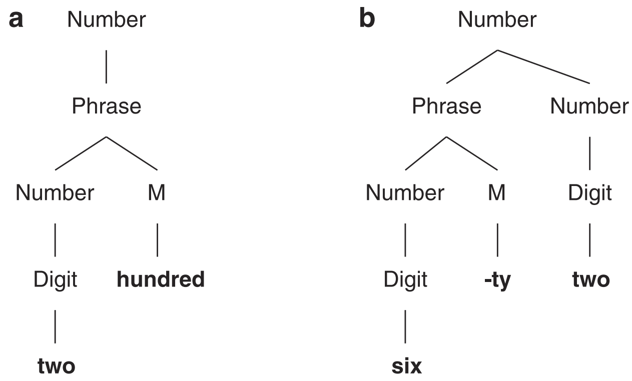


**Fig. 1.** Hierarchical Tree Representation. Note. Hurford's syntactic representation of the cardinal number words (a) two hundred and (b) sixty-two.

corresponds the additive operation 200 + 5. We return to these points later.

Another issue with assuming the recursive successor function in one's mind is that, from a purely computational point of view, this model predicts larger numbers to take more time to represent than small numbers because they require more calls to the recursive function.[4] In other words, thinking about 200 should take much longer than thinking about 20. However, we know of no such demonstrations. Of course, one may argue that recursive successor function is only used to understand the logic behind the natural number system but not for representing specific numerical values. This kind of argument, however, is problematic because it defeats the purpose of the formalism as a basis of a psychological theory.

*Is the knowledge of the successor principle measurable in children?*

A related challenge for the successor principle-based theories is that there does not exist a satisfactory method for unequivocally measuring a child's knowledge about the successor principle of the system of natural numbers. Some authors claim to measure the understanding of the "successor principle" using the Unit task (Cheung et al., 2017; Davidson et al., 2012; Schneider et al., 2021; Spaepen et al., 2018). However, as criticized in the earlier section *Recursive successor function as a product of inductive inference*, performance in the Unit task not only depends on the numbers used in that task but also children's counting skills. For example, passing the Unit task with small $N$ (e.g., 13, 14, or 15) does not guarantee the child to pass the task with larger $N$ (e.g., 24, or 25). Thus, the Unit task at best reveals piecemeal knowledge about how counting works in local number sequences and arguably does not measure children's abstract number knowledge. Furthermore, even if one is to assume that the child uses abstract knowledge about the next number to solve the Unit task, it is impossible to discern if such knowledge is based on recursion or iteration (Luuk & Luuk, 2011, Martins, 2012).

What does the Unit task measure? Theoretically speaking, in order to succeed in the Unit task, a child only needs to know how to infer the next number over the count sequence from one to nine. This is because no matter what number is used, the task only probes about the unit. According to this logic, children should be able to succeed in the Unit task even when the probed numbers are beyond their count list. Such a pattern was indeed found. One study in which children performed the Unit task and a counting task demonstrated that five out of fourteen children who could not count beyond the number 49 were able to solve the Unit task with numbers beyond their count list (e.g., asking for 57 + 1 and 75 + 1) (Guerrero et al., 2020). Given their limited count list, it is likely that those children solved the task with their knowledge that one added to 7 becomes 8 and one added to 5 becomes 6, not because they have a mental representation of the cardinal values 57 and 75, respectively. Even if one assumes that the mental representation of the number 76 comes from $S(75)$, it is unclear if that mental representation is a result of the successive application of the successor function, as in $76 = S(S(S(S(S(S(S(69)))))))$ and eventually $76 = S(\ldots S(1))$, which would be asserted by a successor principle-based theory. As a side note, even adults without formal training in mathematics show little genuine understanding of the successor principle (Relaford-Doyle & Núñez, 2018).

Finally, one may argue that children's explicit knowledge about infinity allows us to tap into children's successor-principle knowledge (Cheung et al., 2017; Chu et al., 2020); however, we disagree. As we explain in the next section, the successor principle is not the only means to understand the generativity of natural numbers, so children's knowledge about infinity need not necessarily arise from the successor principle. Moreover, knowledge about infinity (assuming that it can be measured) at best marks one (near the end) stage of number concept acquisition. Thus, it may not provide essential pieces of information about *how* children get there.

## Then, how do children acquire natural numbers?

So far, we developed an argument that the successor-principle based theories of number acquisition lacks precise explanations

[4] We thank Attila Krajcsi for sharing this idea.

about how children acquire the successor principle. At this point, it is worth questioning whether the idea of the successor principle that has been axiomatic in the number acquisition literature for decades is worth clinging onto. In the following sections, we develop an alternative proposal for explaining children's acquisition of natural number concepts.

*Generative rules of the natural number system*

Let us first be reminded that our definition of generative number concepts refers to the understanding of the rule-governed symbolic system of natural numbers (Wiese, 2003). Thus, the psychological question about how children acquire natural numbers becomes how they understand the rules of this system and which rule or rules they acquire. Obviously, one candidate is the successor principle; however, there exist other mathematical principles that allow general statements that hold for all natural numbers. We argue that the key to answering the aforementioned critical question is constructing the most reasonable hypothesis about which mathematical principle(s) children learn in the process of learning the meaning of numerals and in turn acquiring number concepts.

Here, we develop the argument that children's understanding about the combinatorial rules of the *numerals* give rise to their acquisition of the combinatorial rules of number. In other words, we propose that the structure of numerals enable generative number concepts. Similar arguments are made in Buijsman (2018, 2020) from a philosophical perspective. Prior to introducing the specifics of our proposal, we first introduce a theoretical analysis of numeration systems that illustrate the importance of numerals in understanding the numerical system.

*External representations of number affect numerical thinking*

A seminal work by Zhang and Norman (1995), which is further extended by others (Bender et al., 2015; Bender & Beller, 2011, 2017; Schlimm, 2018), provides an in-depth theoretical analysis about how combinatorial rules of numerals affect the way humans think about number. According to these authors, our understanding of number and ability to solve numerical tasks comes from the integration of distributed numerical information represented externally as numerals and internally in memory. They argue that how number is represented externally has an impact on cognitive processes such as working memory, cognitive load, and attention affecting the learning of numerical information. This framework can well explain how Arabic numerals (or positional numeration systems in general) allow the representation of number in ways that no other numeration systems do (Zhang & Norman, 1995).

Zhang and Norman (1995) developed a hierarchical taxonomy for number representations in four levels: dimensionality, dimensional representation, base, and symbol representation. Dimensionality is categorized broadly into several systems, which differ in efficiency in representing numbers. Of importance for the current discussion is the dimensionality that represents number via one base dimension ($a_i$) and one power dimension ($x^i$) as a polynomial (6) (referred to as the 1 × 1D system in Zhang & Norman, 1995). It is the most widespread dimensionality seen in modern numeration systems (e.g., cardinal number words in English or Arabic numerals). The historical origin of the base dimension is likely to be *sequential ordering* with one-to-one correspondence between distinct numerals and objects (such as the body-part counting system). The historical origin of the power dimension is likely to be *grouping* of supplementary quantities (such as using a big diagonal hash mark to indicate groups of five in a tally system) which is also observed very early in human history (Menninger, 1969).

(6) $\Sigma a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x^0$

Numeration systems with the same dimensionality may have different dimensional representations, and the importance of external representation can be best understood in the analysis at this level. Take Arabic, Chinese, and Egyptian numerals for example. The base and power dimensions in Arabic numerals are represented by shape (0, 1, 2, 3, … 9) and position ($10^n$, $10^{n-1}$, $10^{n-2}$, …, $10^1$, $10^0$), respectively. The two dimensions in Chinese numerals are represented both by shape. The two dimensions in Egyptian numerals are represented by quantity and shape. Importantly, the shape and position of Arabic numerals are externally separable, whereas the all the other numeration systems invented across the world require internal separation of dimensions. The external separation of base and power in Arabic numerals was a consequence of the discovery of zero to mark empty positions (Kaplan, 2000; Seife, 2000), which explains why most lay people were not able to perform simple arithmetic operations (in today's standards) prior to the invention of the Arabic system (Dantzig, 1954). Even the most gifted minds in the ancient world struggled with some number concepts like negative numbers and infinity, which seem today like a natural corollary to the concepts of natural numbers (Ifrah, 2000; Rucker, 2019). In sum, the invention of a new external representation of number (i.e., Arabic numerals) brought a paradigm shift in how humans conceptualize number.

*Syntax of complex number words enables arithmetic thinking*

As introduced in an earlier section, the syntactic structure of cardinal number words (5) enables additive and multiplicative interpretations. Where do those interpretations come from? Anthropological and archaeological record suggests that numerals originate from concrete perception-based supplementary quantities (Ifrah, 2000; Menninger, 1969). These supplementary quantities were often concatenated (e.g., / and / becomes //) or grouped (e.g., ///// becomes ~~////~~) for a more efficient representation of quantities, since a long sequence of tallies could be cumbersome. The remnants of these operations are evident in verbal numerals as well. In many cultures, a sign for a unit of quantity (usually-one or two) was adopted and used in concatenation to indicate a larger quantity, as illustrated by a*mẽmẽ*, a*tuca*, and a*tuca amẽmẽ* in the Aikanã language described earlier. This is a likely basis for how the operation (5b) is interpreted additively. However, additive numeration systems based on a small lexicon are not efficient for representing larger numbers in spoken language because of memory and attentional constraints.

Such an inefficiency can be resolved by grouping supplementary quantities and assigning names to those groups which appeared early in history (Menninger, 1969). The Fiji Islander carved one large notch after every-nine notches into the club and the native warrior in the Philippines inlaid silver nails in groups of three on the sword to count the victims. Grouping also is evident in verbal numerals. In Gothic, for example, 50 (*fimf-tigjus*) is expressed as five tens not five ten. Another example is quatre-vingts (i.e., 80) in French as it literally means four twenties. Similarly, in Spanish, the suffix -es is used for a multiplier to indicate plurality (*un millon* for *one million* but *tres millones* for *three million*). These plural markers suggests that these verbal numerals represent a physical grouping operation. That said, even though the operation (5c) is interpreted as a multiplicative operation ($N_1 \times N_2$), it likely originates from repeated addition ($\sum_1^{N_1} N_2$). In fact, repeated addition replacing multiplication is observed in everyday mathematics even nowadays (Carraher et al., 1985), although distinguishing between these two models is not a focus of this paper. What is clear, however, is that the logical interpretation of the operation (5c) is multiplicative, instead of some other mathematical operations such as the third root of ten or three to the tenth power for [three ten], considering the historical origins of numerical thinking in which grouping played a central role. After all, the syntactic structure of cardinal number words represents additive and multiplicative operations of quantity, which can be formally explained in term of polynomial dimensional representations (6) (Zhang & Norman, 1995). A similar emphasis on addition and multiplication was made in Spelke (2017).

Earlier in the section *External Rrepresentations of number affect numerical thinking*, we described how Arabic numerals as a positional system allow an entirely different conceptualization of natural numbers compared to all other numeration systems. In practice, Arabic numerals make complicated arithmetic much easier (Zhang & Norman, 1995). Considering the way in which natural numbers are conceptualized, the positional numeration system seems superior in such a task. However, this is not to say that other non-positional numeration systems are inferior overall (Chrisomalis, 2020). For example, a training study demonstrated evidence that simple comparison and addition were easier with an artificial quantity- and shaped-based numeration system (akin to Roman numerals) than with an artificial positional numeration system (Krajcsi & Szabó, 2012). Likewise, for an intuitive representation of magnitude, cardinal number words in theory should be superior to Arabic numerals. This is because the structure of verbal numerals could be mapped to physical operations on sets, while the structure of Arabic numerals does not. For instance, just as the mental model of seeing both a {cat} and a {dog} would be {cat, dog}, the mental model of seeing 1 group of 10 items and 7 items together would be {ten, seven} consistent with the rules of numerical syntax. In contrast, the same operation using Arabic numerals would result in the wrong notation {107}. In fact, such an erroneous response pattern was found in previous work on numerical transcoding (Barrouillet et al., 2004; Byrge et al., 2014; Granà et al., 2003; Hurtado et al., 2010; Moeller et al., 2015; Noël & Seron, 1995; Pourquié & Nespoulous, 2018; Power & Dal Martello, 1990).

In sum, while both cardinal number words and Arabic numerals are based on polynomial representations (6), cardinal number words enable more intuitive representations of addition and multiplication thanks to the explicit syntactic structure allowing those arithmetic interpretations. These properties are critical for the generative rules of the natural number system, as we elaborate in the following section.

## Theoretical proposition: Arithmetic representations of cardinal number words enable the concept of generative algebraic properties of natural numbers

Natural numbers are defined within an abstract symbolic system with generative rules. Thus, we believe that acquiring number concepts is equivalent to understanding the generative rules of the natural number system. Most, if not all, psychological theories of number acquisition postulate the successor principle (1–3) as the generative rule that children eventually acquire, which perhaps reflects the influence of logico-mathematical definition of natural number (Decock, 2008). Of course, there is no doubt that the knowledge about a well-ordered sequence (that is, "successive" nature) of number words is necessary for the acquisition of number concepts, as theorized in previous accounts (Carey, 2004; Gelman & Gallistel, 1978). We have reviewed several pieces of evidence regarding the importance of children's understanding of a *local* sequence of number words. However, our analysis of the theories and the literature indicates a large explanatory gap in exactly how the knowledge about number sequence relates to the acquisition of the successor principle. We also question if the knowledge about the successor principle is even empirically measurable.

We propose that children's acquisition of generative number concepts in modern society is essentially them coming to implicitly understand the arithmetic properties of natural numbers, which are enabled by the additive and multiplicative representations of complex number words. Addition and multiplication in natural numbers have important mathematical properties. Most crucial for our proposal is the property of closure: for all natural numbers $N_1$ and $N_2$, both $N_1 + N_2$ and $N_1 \times N_2$ are always natural numbers. Theoretically, addition and multiplication operations allow one to conceptualize all and only the members of the abstract system of natural numbers. Such a property is crucial because generative number concepts are defined as the understanding about the rule-governed system that indefinitely generates all and, importantly, only the natural numbers. According to this logic, subtraction or division would not work as the rules of the system because they yield negative numbers and fractions which are beyond the boundary of natural numbers. We propose that the conceptualization of these properties comes from the understanding about how the structure of complex number words allows the interpretations of additive and multiplicative operations, which are rooted in concatenation and grouping of physical objects. Note that complex number words are concrete and finite, and physical objects that may be used for arithmetic operations are concrete and finite (McCrink & Wynn, 2004). Hence, the analogical mapping (see Gentner, 2010) between the two types of external representations enables the emergence of the abstract, internal representations of natural number, and eventually algebraic thinking, in children.

Our proposal paves way for new research directions in the acquisition of generative number concepts. Specifically, formalized rules of numerical syntax by Hurford (1975) and formalized representations of modern numeration systems by Zhang & Norman (1995) provide ways to construct empirical questions and testable hypotheses regarding children's understanding of the external representations of number. The following section provides novel ways to understand previous findings as well as novel hypotheses that stem from our theoretical framework.

*Hypotheses stemming from our theoretical framework*

Rules of numerical syntax highlight two important properties that determine one's knowledge about the numerical meaning behind cardinal number words. The first is the sequential structure of the lexical category Digit triggered by counting, and the second is the embedded structure of complex number words that allow arithmetic interpretations. Children's acquisition of numerical syntax, therefore, must involve both properties. However, there is no a priori reason to believe that the two properties are dependent on each other, neither for comprehension nor production. Such an idea raises the following hypotheses.

1) Learning the sequential structure of Digit and the embedded structure of complex number words could be independent processes.
2) Children could use their knowledge about these two properties to reason about properties of number beyond their counting ability.

Our proposal that children acquire generative number concepts via understanding numerical syntax leads to several additional hypotheses.

3) Children rely on these two properties to solve additive problems.
4) The structure of numerical syntax in a language influences children's understanding of number.
5) Complex number words are represented as a hierarchical structure.

An essential property of the set of natural numbers is that they do not have an upper limit. Children are thought to acquire this idea when they realize that numbers are potentially infinite. While some authors assume that children's knowledge of infinity rest on the successor function (Cheung et al., 2017; Chu et al., 2020), an alternative hypothesis can be developed from our perspective.

6) Children's knowledge about infinity is derived from generative rules that depend on the structure of numerals (i.e., numerical syntax and positional representations).

Finally, our proposal makes an emphasis on the role of numerals (both verbal and Arabic) in the acquisition of number concepts and thus offer the following hypotheses.

7) Numerical syntax dominates children's early representations of number because it makes additive and multiplicative operations explicit, whereas the place value system does not.
8) Children's knowledge about numerical syntax underlies the acquisition of the polynomial dimensional representations of large numbers.
9) Mature understanding of the properties of number is facilitated as children learn to effectively use multiple representations (i.e., both numerical syntax and place value).

Below we provide empirical evidence that supports each of these hypotheses.

**Hypothesis 1**. *Learning the sequential structure of Digit and the embedded structure of complex number words could be independent processes*.

Our proposal that children learn generative number concepts partly via numerical syntax predicts that learning the numerical syntactic categories, Digit and Multiplier, can occur in parallel and independently of each other. In other words, children do not necessarily need to acquire the meaning of all the Digits to understand the meaning of Multipliers, unlike what a successor principle-based theory would predict.

Cheung et al. (2016) explored the precursors for understanding Multipliers. Using an artificial base-three numeration system, they demonstrated how Chinese-speaking and English-speaking children (4–6 years) could acquire and use a new Multiplier productively. During the modeling phase, the researcher showed a picture of a set of three items (e.g., houses) and used the pseudoword "gobi" that represented a novel Multiplier (sets of three) to describe the picture (e.g., There are one gobi houses). During the training phase, the researcher showed two pictures, one showing three items and the other showing a quantity other than three, and children were asked to choose the picture showing one gobi items. In this training phase, children obtained corrective feedback. The test phase evaluated if the children were able to generalize the use of the novel Multiplier. The procedure was identical to that in the training phase, except in this phase one of the quantities was six, and the researcher asked, "Who has two gobi houses?" The results showed that more than half of the children performed above chance. However, neither children's age nor their counting knowledge predicted the generalization capacity. These results indicate that children understand how Multipliers work at a very young age, irrespective of their sequential counting ability.

This hypothesis also allows us to make related novel predictions about how children acquire the meaning of Multipliers. For instance, children may go through the following stages. First, at a very young age, children may not understand the compositionality of complex number words, meaning not able to distinguish Digits from Multipliers. Second, children may understand that Multiplier is an indicator of the exact size of a set (although perhaps without knowing what quantity maps to each Multiplier) and understand the ordinal relations between some Multipliers (see Cheung & Ansari, 2022). Finally, they may come to understand that Multipliers indicate the size of a unique exact quantity.

**Hypothesis 2.** *Children could use their knowledge about these two properties to reason about properties of number beyond their counting ability.*

Some support for this hypothesis is already discussed earlier in this paper. When asked to count as high as they can, children often stop at the end of a decade. However, with some external help to produce the next decade, many children can count higher than their initial highest count (Chu et al., 2020; Schneider et al., 2021). These findings illustrate that children use their understanding about the regularities in complex number words to rote count beyond their ability.

Similarly, children use numerical syntax to recognize the place value structure of large numbers. In the Number-Multiplier Syntax task administered in Guerrero et al. (2020), children were given an open abacus with three columns representing the power dimension (ones, tens, and hundreds positions). The experimenter put an equal number of beads in the ones and tens columns and said the number represented in the abacus to the child (e.g., forty-four). Then, the experimenter put one bead in the position of ones or tens columns and asked a forced-choice question (e.g., "Is the new number 45 or 54?"). The same procedure was followed using ones, tens, and hundreds columns (e.g., the beads represent five hundred fifty-five), adding one bead in tens or hundreds columns, and asking a forced-choice question (e.g., "Is the new number 454 or 544?"). Children with limited counting knowledge nevertheless were able to solve these tasks.

Reasoning about number properties in a similar manner is observed in other tasks as well. As mentioned earlier, children are shown to solve the Unit task beyond their counting ability (e.g., solving for 57 + 1 or 75 + 1 when they cannot count beyond 49) (Guerrero et al., 2020). Unpublished work in progress from our lab also seems to suggest that children can compare numbers beyond their counting ability presumably using their knowledge about the structure of complex number words. In this study, children compared two complex number words that differed only in one syntactic position (e.g., *five* hundred thirty boxes versus *eight* hundred thirty boxes) and were asked to choose which is more. On average, children succeeded in this task even though these numbers were beyond their count range. One may argue that the children may not have had to represent the embedded structure of number words and rather could have solved the task by simply comparing one word at a time. However, our preliminary data suggest that this explanation is unlikely because children's performance was modulated by the complexity of the numerical syntax. For instance, children found it more difficult to compare numerals with a multiplicative merge structure (e.g., *five* hundred thousand boxes versus *eight* hundred thousand boxes) than numerals with an additive merge structure (e.g., *five* hundred thirty boxes versus *eight* hundred thirty boxes).

Together, these results suggest that the knowledge about the sequential and embedded structure of number words could be used to address problems about larger numbers that children cannot independently produce.

**Hypothesis 3.** *Children rely on these two properties to solve additive problems.*

Our proposal is consistent with the findings showing that young children rely heavily on the ordinal structure of the count sequence to solve simple arithmetic problems (Carpenter & Moser, 1982; Fuson, 1982; Groen & Parkman, 1972; Hitch et al., 1983; Krebs et al., 2003; Mulhern & Budge, 1993; Secada et al., 1983). When given an addition problem, a typical five-year-old children count all the items to reach the sum. For instance, in the problem asking for five plus three, children would count the number of items (or fingers) referring to the augend (i.e., one, two, three, four, five), then count the number referring to the addend (i.e., one, two, three), and finally count all the items counted (i.e., one, two, … , eight) to reach the solution. This is referred to as the counting-all strategy. In contrast, older children use one of the two numbers as the starting point (e.g., five) and count the number of times represented by the other number (e.g., six, seven, eight) to reach the solution. This behavior is referred to as the counting-on strategy. Later, some children and adults develop a decomposition strategy for numbers represented by complex number words. For example, in a problem asking eighteen plus five, children decompose the latter into two and three, then add the number two to eighteen to get twenty and add the remaining three to reach twenty-three.

While all the strategies illustrate that children rely on counting to solve arithmetic problems with cardinal values, the counting-all strategy demonstrates that children do not fully understand the cardinal value of the augend or summand at this developmental stage. Instead, they rely on the entire count sequence, starting from one, to solve the problem. The decomposition strategy suggests that children and adults rely on the embedded structure of the complex number words using the decade transitions (e.g., eighteen plus two equal twenty) as a subgoal in the process to build the whole number. The transition from counting-all to counting-on strategies also highlights that the sophisticated use of the count sequence coincides with the development of number knowledge. Furthermore, the jump from counting-on to the decomposition strategy shows how the base 10 number system impacts children's comprehension of addition.

More recently, Pinheiro-Chagas et al. (2017) found similar patterns in adults. In their study, participants were asked to solve single-digit addition and subtraction problems on a digital tablet on which participants moved their finger from a starting point to a horizontal number line to indicate the calculated outcome of the given arithmetic problem. Finger movement time was measured to explore the covert processing stages underlying these arithmetic calculations, and indeed it was modulated by the size of the operand with the lower value. This effect is equivalent to the counting on strategy found in children and suggests that adults are no exceptions to the influence of the sequential structure of cardinal numbers on arithmetic problem solving.

**Hypothesis 4**. *The structure of numerical syntax in a language influences children's understanding of number.*

As mentioned in earlier sections, cross-linguistic studies demonstrate that the structure of numerical syntax influences the speaker's comprehension of number (Fuson et al., 1982; Lefevre et al., 2002; Leybaert & Van Cutsem, 2002; Miller & Stigler, 1987; Moeller et al., 2015), in line with our theoretical proposal. However, because most verbal numeration systems follow the base-10 structure, those studies are limited to assessing the effects of the transparency of the base-10 structure in cardinal number words. In another study, how the base structure itself influences children's numerical thinking was tested (Guerrero et al., 2013). In that study, the authors compared arithmetic problem solving performance between Spanish-speaking children and Deaf children in Colombia who use Colombian Sign Language (CSL) as their native language. CSL is a sub-base 5 numeration system, in which the first morphologically complex number word occurs at 6 which is represented using the sign of 1 (i.e., index finger) with a flexion of it. The first morphologically complex number word in Spanish occurs at dieciseis (sixteen), in which the prefix diece is derived from diez (ten) and the suffix seis corresponds to six. The authors found that first-grade Deaf signers perform better than first-grade hearing children in verbal arithmetic problems that include the number five (e.g., $5 + 8$ or $5 + 7$). However, hearing children are more accurate in problems below the number ten (e.g., $3 + 4$ or $3 + 6$). Similarly, in a different study, Guerrero et al. (2018) compared first-grade Deaf children with two groups of first-grade hearing children in a number comprehension task and the highest count task. In the comprehension task, children were asked to say or sign the number represented in tokens representing a sub-base 5 structure. For example, the number 6 was represented using a blue square representing five and a red square representing 1. The results showed that Deaf children were better at naming the number represented in tokens than hearing children, even when hearing children were able to count higher. These findings, together with cross-linguistic studies in rote counting, demonstrate that the transparency and the structure of the numerical syntax influences children's acquisition of number knowledge.

**Hypothesis 5**. *Complex number words are represented as a hierarchical structure.*

One important premise of our proposal is that complex number words are hierarchically represented in the mind. Consistent with this idea, previous studies indeed show that the structure of cardinal number words influences behavioral performance and neural responses in a variety of numerical tasks (Barrouillet et al., 2004; Guerrero et al., 2014; Hung et al., 2015; Lochy et al., 2002). For example, in a transcoding study, Lochy et al. (2002) asked French-speaking university students to write down Arabic numerals after they listened to dictated cardinal number words. The structure of the number words was classified as multiplicative (e.g., douze cents, twelve hundred in English) or additive (e.g., mille deux cents, one thousand two hundred). When the authors analyzed response times for writing each of the Arabic digits, the results showed that the first inter-digit latency (e.g., time interval between the completion of 1 and the initiation of 2 when dictating 1200) was longer in the additive structure (e.g., one thousand two hundred) than in the multiplicative structure (e.g., twelve hundred). These results suggest that the structure of complex number words influence the representation of natural numbers. In the multiplicative structure, the first word corresponds to a lexical primitive (e.g., douze, twelve); therefore, its representation must not reflect an operation. In the additive structure, the first inter-digit latency reflects a merge operation in numerical syntax.

In an fMRI study, Hung et al. (2015) examined brain activities of French- and Mandarin-speaking adults in response to visually-presented complex number words with different hierarchical structures. In both languages, the authors presented complex number words categorized into four conditions that differed in the depth of Merge operations. For instance, in French, the sequence six cent soixante deux mille neuf cent quarante sept build a numeral with four levels of Merge operations that represents the number 662,947. In contrast, the sequence soixante cent neuf six huit cinq trente cent contain no or only a few Merge operations. The results showed that activation in the left inferior frontal gyrus and the left inferior parietal lobule was correlated with the depth of Merge operations. The left inferior frontal gyrus has shown to be associated with syntactic processing and hierarchical structure building in recent neurolinguistic studies (Brennan et al., 2012, 2016; Nelson et al., 2017; Pallier et al., 2011; Tanaka et al., 2019). The fact that the brain regions for representing complex number words overlap with the regions posited for hierarchical structure building is consistent with the idea that complex number words are represented in a hierarchical manner.

**Hypothesis 6**. *Children's knowledge about infinity is derived from generative rules that depend on the structure of numerals* (*i.e.*, *numerical syntax and positional representations*).

Some recent studies have explored children's comprehension of infinity with an idea that infinity is a product of the successor principle. Cheung et al. (2017) and Chu et al. (2020), for instance, inquired children's knowledge about infinity by asking them about the biggest number they know and about adding one to that biggest number. They classified those as an infinity knower who explicitly verbalized that there was no biggest number and that it is always possible to keep adding 1 to a number. However, it is unclear how such knowledge informs one's understanding about the successor principle. In theory, the successor function is an abstract recursive rule, while adding 1 is an operation between cardinal values; whether they engage the same mental process is worth questioning. In practice, evidence shows that children generate a new number based on numerical syntax, not based on the $N + 1$ rule. In order to test the concepts of infinity in children, Falk (2010) devised interview-based games that encouraged children to generate a large number. In one game, the children were told "Now let's say larger and larger numbers by turns. I start with [e.g.] 10." A 9-year-old (CH) in this game produced the following conversation with the researcher (R).

R: 500
CH: 600
R: 20,000

CH: 30,000
R: million
CH: 2 millions
R: billion
CH: What is that?
R: A very large number
CH: Aha! [ironically] 2 billions
R: trillion
CH: 2 trillions
R: googol
CH: 2 googols
R: Will the numbers end up for us?
CH: No, because each time you make up a number, I can add to it.
R: Did you recognize these numbers?
CH: No.
R: How did you manage to find bigger numbers?
CH: Because you said.

As shown, the child in most cases simply changed the first Digit even while enduring an ungrammatical numeral structure (e.g., two millions). Note that increasing the first Digit of a complex number word by one cannot be characterized as the use of the successor function because the successor principle by definition does not assume any hierarchical or embedded structure. Instead, the child's response pattern should be characterized as the usage of the local sequence of Digit in numerical syntax. Such a pattern is consistent with the prediction that children's knowledge about unboundedness and infinity of numbers primarily arises from the generative structure of numerical syntax (also see Buijsman, 2018). Based on this hypothesis, we also predict unique response patterns when children were given the same task using Arabic numerals (e.g., adding more digits to generate a bigger number). Of course, there could be individual, cultural, and developmental differences in how children and adults conceptualize infinity. Some populations may rely more on one aspect of the structure of numerals than another (e.g., additive over multiplicative operations or vice versa) to conceptualize infinity. It is also possible that some populations may rely on the abstract successor function to conceptualize infinity, although we hypothesize against this possibility in children. Characterizing these individual differences would be a fruitful line of future research.

**Hypothesis 7**. *Numerical syntax dominates children's early representations of number because it makes additive and multiplicative operations explicit, whereas the place value system does not.*

As mentioned, transcoding studies from number words to Arabic numerals show that numerical syntax heavily influences one's representation of number and that the place value rules are much more challenging to learn (Granà et al., 2003; Herzog & Fritz, 2022; Hurtado et al., 2010; Noël & Seron, 1995; Power & Dal Martello, 1990). In such a transcoding study, for instance, one of the most frequent errors corresponds to the literal representation of the components of the number word. An error of this type is writing "20030" for the number word two hundred thirty, even though those who commit this type of error tend to correctly transcode 200 for two hundred and 30 for thirty.

Transcoding from Arabic numerals to cardinal number words also demonstrate the influence of numerical syntax on children's numerical representations. Earlier studies have shown that children as young as three years of age are able to, at the above-chance level, map spoken number words to written multi-digit Arabic numerals and compare their magnitudes (Mix et al., 2014; Yuan et al., 2019). In a more recent study by Vasilyeva et al. (2022), Russian preschoolers' error patterns when reading aloud multi-digit Arabic numerals were examined. In that study, many of children's number naming errors corresponded to syntactic errors including substitution (saying "sixty-seven" for 97), reversal (saying "seventy-nine" for 97), and place-value marker error (saying "nine-hundred seven" for 97), suggesting that children are strongly influenced by numerical syntactic rules and that learning place value is a big challenge for children (e.g., Cobb & Wheatley, 1988; Fuson & Briars, 1990). Interestingly, the frequency of syntactic errors decreased with age and was negatively associated with children's base-10 representations measured by a base-10-block task (Vasilyeva et al., 2022). These results suggest that children's representation of number becomes more flexible as they become capable of using both numerical syntax and place value to represent the base-10 structure.

**Hypothesis 8**. *Children's knowledge about numerical syntax underlies the acquisition of the polynomial dimensional representations of large numbers.*

In a recent study, Guerrero et al. (2020) investigated children's knowledge about numerical syntax and tested how this knowledge relates to the understanding about the meaning of complex number words. In that study, children's understanding about the embedded nature of the meaning of complex number words was tested using the novel Give-a-number Base-10 (Give-N10) task. In that task, children were asked to retrieve items in a requested quantity expressed in complex number words such as 16, 27, and 32. Importantly, the retrieval of these items was modeled by the experimenter who retrieved a group of ten items at a time to signify the embedded base-10 structure (i.e., polynomial dimensional representations) of complex number words. Children's performance was scored based on how they used the grouping strategy to retrieve 16, 27, and 32 items. The results showed that, on average, it was not until six years of age when children successfully performed this task using the base-10 structure. Also, children's Give-N10 performance, while controlling for school grade, was strongly predicted by their performance in two other tasks that assessed children's numerical syntax knowledge. These findings indicate that children's knowledge about numerical syntax seems to be related to a task designed to tap into their understanding of the polynomial dimensional representations of large numbers. The results also suggest that Give-N10 is a reasonable candidate task for assessing children's knowledge about the arithmetic meaning embedded in complex number words.

**Hypothesis 9**. *Mature understanding of the properties of number is facilitated as children learn to effectively use multiple representations* (*i.e.*, *both numerical syntax and place value*).

Even though numerical syntax is critical for children's understanding of large numbers, their knowledge about Arabic numerals and the flexible transcoding between verbal and Arabic numerals are known to be important predictors of their mathematical thinking. For example, numerous previous studies have demonstrated that transcoding ability is crucial for the development of arithmetic skills even when well-known numerical predictors of math achievement are considered (Banfi et al., 2022; Göbel et al., 2014; Moeller et al., 2011; Moura et al., 2013; van der Ven et al., 2017). An important question that remains to be addressed is how children's understanding about Arabic numerals contributes to their reasoning about the natural numbers.

Based on our theoretical proposal, a novel hypothesis concerning this underlying mechanism can be constructed: it is because the place value system complements numerical syntax by making the combinatorial and algebraic relations between numbers explicit (Zhang & Norman, 1995). Take some of the simplest addition cases for instance. The operation needed to perform two hundred plus two hundred using verbal numerals is quite different from the operations needed to perform two hundred plus twenty. In the first case, the multiplier hundred should be retained while the two digits two and two are added. In the second case, the two numbers should be simply merged. That is, even in such simple cases, the operations required to obtain the results are different. In contrast, the problems 200 + 200 and 200 + 20 using Arabic numerals require a single set of logical operations (e.g., 2 + 0 = 2, 2 + 2 = 4) along with the rules of positional notation to solve correctly. Furthermore, numerical operations that require the introduction of a new power dimension are challenging with verbal numbers (e.g., *ninety* plus *twenty* is not *eleventy*). In contrast, such operations are straightforward with Arabic numerals; a single set of logical operations, 9 + 2 = 11 and 0 + 0 = 0, allows the computation of 90 + 20, 900 + 200, 9000 + 2000, etc.

Finally, according to our proposed framework, number acquisition is a process of understanding the polynomial representations of number allowed by numeration systems and eventually the algebraic system of natural numbers. Thus, the base system serves a critical role for number acquisition. Most modern numeration systems use the main base of 10. However, as mentioned, some languages additionally employ a sub-base system (e.g., sub-base 5 in Colombian Sign Language) and many ancient cultures used different main bases, such as 60 in Babylonian and 20 in Aztec and Mayan. Given the same range of numbers to be represented, the size of the base dimension and the size of the power dimension must be inversely related (6). Hence, it is reasonable to assume that there is an optimal base size for one's cognitive capacity. Although it would be difficult to test empirically, an interesting thought experiment would be to test how the size of the base dimension influences the speed or ease of number acquisition.

## Conclusion

How children conceptualize natural numbers has been a central cognitive and developmental science question for decades. While the idea that the successor principle is the destination of number acquisition has been axiomatic (Carey, 2004), our critical analysis of existing psychological theories undermines this idea. Certainly, the successor principle is one way to define and conceptualize natural numbers theoretically, but we identified a large explanatory gap in the mechanism for how children acquire it. We first criticized the vague use of the term successor principle in the field; it is used to describe both successive mapping between objects and count labels as well as the abstract generalized knowledge about the system of natural numbers. We also argued that there is little theoretical basis for how understanding the regularities in the local count sequence or even the recursively defined numerical syntax is linked to the acquisition of the recursively defined successor principle, because the two forms of recursion are distinct. Finally, we criticized the lack of empirical methods for evaluating children's knowledge about the abstract successor principle. Hence, one related challenge to the successor-principle-based theories is that they are largely unfalsifiable.

In this paper, we developed an alternative theoretical proposal that one culminating point of children's number concept acquisition is marked by their acquisition of the generative arithmetic properties of natural numbers, which is primarily enabled by the additive and multiplicative properties inherent in the structure of complex number words. Importantly, under addition and multiplication, natural numbers are mathematically closed, allowing the indefinite generation of all and only the natural numbers. Understanding these properties need not be explicit at first. In fact, it is more likely that children's understanding starts as an implicit and incomplete knowledge because number words as a non-positional numeration system impose practical limitations for arithmetic representations and operations in full capacity. Later in the acquisition path at least in modern society, children's algebraic thinking becomes more explicit and precise as they learn the positional system (e.g., Arabic numerals) which facilitates polynomial representation of natural numbers.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

*Acknowledgements*

We appreciate Tom Roeper, Erik Cheries, and Stefan Buijsman for helpful discussions and comments on early versions of this manuscript. We also appreciate David Geary and Attila Krajcsi for their invaluable comments and suggestions. Finally, we are grateful to Jenna Croteau for inspiring discussions and constructive feedback for improving our idea and the paper.

*Funding statement*

This work was supported by the Fulbright-COLCIENCIAS Fellowship to D.G. and by the National Science Foundation CAREER award BCS1654089 to J. P.

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