Background Field Method

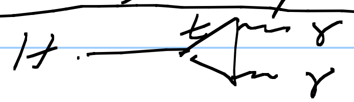
Note Title 7/17/2018

Useful in forming and renormalizing EFTs

Also other calculations

Keep external fields in calculations

Fun example $GG \to H$ and $H \to \gamma\gamma$ (approx if $M_t \gg M_H$)

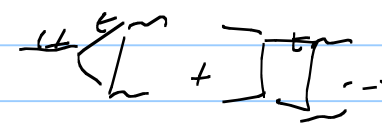


$$\mathcal{L}_t = -\frac{\Gamma_t}{\sqrt{2}}(v+H)\bar{t}t = -m_t\left(1+\frac{H}{v}\right)\bar{t}t = -m_t(H)\bar{t}t$$

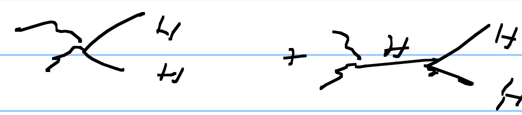
Calculate $\sim O \sim = \left[\ldots\ldots - \frac{e^2}{12\pi^2}\ln\frac{M_t^2(H)}{\mu^2}\right](g_{\mu\nu}q^2 - q_\mu q_\nu)$

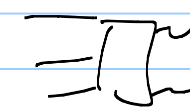
$$\leftarrow \ln \frac{M_t^2}{\mu^2} + \ln\left(1+\frac{H}{v}\right)^2$$

$$\mathcal{L}_{eff} = \frac{\alpha}{18\pi}\ln\left(\frac{v+H}{v}\right)F_{\mu\nu}F^{\mu\nu} + \frac{\alpha_s}{12\pi}\ln\left(\frac{v+H}{v}\right)F^{a\mu\nu}F^a_{\mu\nu}$$



HW show cancelation in $GG \to HH$ at threshold





Example #2 QED with massless scalar

not so useful here, but makes contact between formalism and usual calculations

$$\mathcal{L} = (D_\mu \phi)^* (D_\mu \phi) - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \quad ; \quad D_\mu = \partial_\mu + i A_\mu$$

$$- \phi^* D_\mu D^\mu \phi \quad , \quad D_\mu D^\mu = \Box + i \{\partial_\mu, A^\mu\} - A_\mu A^\mu$$

$$= \Box + V$$

Path Int

$$\int d\phi d\phi^* \, e^{i \int d^4x \, \phi^* D^2 \phi} = N e^{-\mathrm{Tr}(\ln D^2)} = N e^{-\int d^4x \langle x | \ln D^2 | x \rangle}$$

$$\ln D^2 = \ln(\Box + V(A)) = \underbrace{\ln \Box}_{\text{drop}} + \ln\left(1 + \frac{1}{\Box} V\right)$$

$$= \frac{1}{\Box} V - \frac{1}{2}\frac{1}{\Box} V \frac{1}{\Box} V + \cdots$$

$$\langle x | \frac{1}{\Box} | y \rangle = D_F(x-y)$$

$$\langle x | \frac{1}{\Box} V | x \rangle =$$  $$D_F(x-x) V(x) \rightarrow 0$$

Second order piece

$$\frac{1}{2} \mathrm{Tr} \frac{1}{\Box} V \frac{1}{\Box} V = \frac{1}{2}\int d^4x \int d^4y \, D(x-y)\, V(y)\, D(y-x)\, V(x)$$

$$\Delta \mathcal{L} = \int d^4x \, d^4y \; F_{\mu\nu}(x) \frac{D_F^2(x-y)}{4(d-1)} F^{\mu\nu}(y)$$

$$D_F^2(x-y) = F.T.\left[\underbrace{\frac{-i}{16\pi^2}\left(\frac{1}{\varepsilon} + \cdots\right)}_{\text{renorm of field}} + \ln \frac{q^2}{\mu^2} \right]$$

$L(x-y) = FT \ln q^2$

$$\Delta \mathcal{L} = \frac{-i}{16\pi^2}\frac{1}{\varepsilon} \int d^4x \, F_{\mu\nu} F^{\mu\nu} + b e^2 \int d^4x \, d^4y \, F_{\mu\nu}(x) \, L(x-y) \, F^{\mu\nu}(y)$$

General

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \qquad d_\mu = \partial_\mu + \Gamma_\mu$$

$$\mathcal{L} = \phi^+ \left[d_\mu d^\mu + \sigma(x) \right] \phi$$

$$\Delta \mathcal{L}_{div} = \int d^4x \, \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} \cdots \right] \mathrm{Tr} \left[\frac{1}{12} [d_\mu, d_\nu][d^\mu, d^\nu] - \frac{1}{2}\sigma^2 \right]$$

Back to σ model

$$\mathcal{L} = \frac{\nu^2}{4} \mathrm{Tr}(\partial_\mu U \, \partial^\mu U^+)$$

Path $\quad U = \bar{U} e^{i \vec{\tau} \cdot \vec{\Delta}^a}$

$\uparrow$ B.F $\qquad \nwarrow$ fluct.

$$\mathcal{L} = \mathcal{L}(\bar{U}) + \Delta^a [d_\mu d^\mu + \sigma] \Delta^a$$

$\nwarrow \; d = \partial_\mu + \Gamma_\mu(\bar{U})$

$$\Rightarrow \Delta\mathcal{L} = \mathrm{Tr}\left\{ \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{2}\sigma(x) \right\} = \text{Local}$$

$$\Rightarrow \ell_1^r \;,\; \ell_2^r$$

Appendix B

Advanced field theoretic methods

B–1 The heat kernel

When using path integral techniques one must often evaluate quantities of the form

$$H(x,\tau) \equiv \left\langle x \left| e^{-\tau\mathcal{D}} \right| x \right\rangle \quad , \tag{1.1}$$

where $\mathcal{D}$ is a differential operator and τ is a parameter. In this section, we shall describe the *heat kernel* method by which $H(x,\tau)$ is expressed as a power series in τ. For example, if in d dimensions the differential operator $\mathcal{D}$ is of the form

$$\mathcal{D} = \Box + m^2 + V \ , \tag{1.2}$$

where V is some interaction, then the heat kernel expansion for $H(x,\tau)$ is

$$H(x,\tau) = \frac{i}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \dots\right] \ . \tag{1.3}$$

where $a_n(x)$ are coefficients which will be determined below.

$$\langle x|\ln \mathcal{D}|x\rangle = -\int_0^\infty \frac{d\tau}{\tau}\,\langle x\left|e^{-\tau\mathcal{D}}\right|x\rangle + C \ , \tag{1.6}$$

where C is a divergent constant having no physical consequences. Substituting Eq. (1.3) into the above yields

$$\langle x|\ln \mathcal{D}|x\rangle - C = -\frac{i}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} m^{d-2n}\ \Gamma\left(n-\frac{d}{2}\right) a_n(x) \quad . \tag{1.7}$$

$$\mathcal{D} = d_\mu d^\mu + m^2 + \sigma(x) \qquad \left(d_\mu \equiv \frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)\right) ,$$

$$a_0(x) = 1 \ , \qquad\qquad a_1(x) = -\sigma \ ,$$
$$a_2(x) = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}\,[d_\mu,[d^\mu,\sigma]] \quad .$$