

Quantum Corrections to the Reissner-Nordström and Kerr-Newman Metrics

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Abstract

We use effective field theory techniques to examine the quantum corrections to the gravitational metrics of charged particles, with and without spin. In momentum space the masslessness of the photon implies the presence of nonanalytic pieces $\sim \sqrt{-q^2}, q^2 \log -q^2$, etc. in the form factors of the energy-momentum tensor. We show how the former reproduces the classical non-linear terms of the Reissner-Nordström and Kerr-Newman metrics while the latter can be interpreted as quantum corrections to these metrics, of order $G\alpha\hbar/mr^3$.

1 Introduction

The gravitational field around a particle is described by a metric, which provides a solution to Einstein's Equation of general relativity. This theory is clearly a classical field theory. In this paper, we will discuss quantum corrections to the metric and show that these quantum effects are reliably calculable for the case of a charged particle, either with or without spin, using the techniques of effective field theory[1, 2, 3, 4].

The classical solution of the Einstein equations for a massive charged field are described by the Reissner-Nordström metric. In harmonic gauge, this metric has has the form¹[5]

$$\begin{aligned}
 g_{00} &= 1 - \frac{2Gm}{r} + \frac{G\alpha}{r^2} + \dots \\
 g_{0i} &= 0 \\
 g_{ij} &= -\delta_{ij} - \delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^2} + \dots
 \end{aligned} \tag{1}$$

The metric for a spinning charge is known as the Kerr-Newman metric and its form in harmonic gauge is found to be[6, 7]

$$\begin{aligned}
 g_{00} &= 1 - \frac{2Gm}{r} + \frac{G\alpha}{r^2} + \dots \\
 g_{0i} &= \left(\frac{2G}{r^3} - \frac{G\alpha}{mr^4} \right) (\vec{S} \times \vec{r})_i \\
 g_{ij} &= -\delta_{ij} - \delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^2} + \dots
 \end{aligned} \tag{2}$$

We will use effective field theory techniques to recreate both the classical terms in these potentials, and also find quantum corrections of the following forms. For the Reissner-Nordström metric we find

$$\begin{aligned}
 g_{00} &= 1 - \frac{2Gm}{r} + \frac{G\alpha}{r^2} - \frac{8G\alpha\hbar}{3\pi m r^3} + \dots \\
 g_{0i} &= 0 \\
 g_{ij} &= -\delta_{ij} - \delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^2} + \frac{4G\alpha\hbar}{3\pi m r^3} \left(\frac{r_i r_j}{r^2} - \delta_{ij} \right) + \dots
 \end{aligned} \tag{3}$$

¹Note that throughout this paper we use $\alpha = e^2/4\pi$ without any factor of \hbar so that we may use it to describe the classical corrections. This does not make a difference in the sections where we use relativistic notation with $\hbar = 1$, but we adopt this convention so that the all factors of \hbar will be visible in those equations where we make this constant explicit. We use $c = 1$ units in all sections.

For spin 1/2 particle we reproduce the Kerr-Newman metric with $J = \hbar/2$ plus quantum effects of the form

$$\begin{aligned}
g_{00} &= 1 - \frac{2Gm}{r} + \frac{G\alpha}{r^2} - \frac{8G\alpha\hbar}{3\pi m r^3} + \dots \\
g_{0i} &= \left(\frac{2G}{r^3} - \frac{G\alpha}{m r^4} + \frac{2G\alpha\hbar}{\pi m^2 r^5} \right) (\vec{S} \times \vec{r})_i \\
g_{ij} &= -\delta_{ij} - \delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^2} + \frac{4G\alpha\hbar}{3\pi m r^3} \left(\frac{r_i r_j}{r^2} - \delta_{ij} \right) + \dots \quad (4)
\end{aligned}$$

Notice that the spin-independent terms are universal for both the classical and quantum corrections.

We start by discussing the general concept of quantum effects in the metric. To the best of our knowledge the content of this discussion has not appeared before in the literature. We review the logic of effective field theory that we use to analyse this problem. We then turn to the extraction of the specific quantum corrections that are relevant for this problem, first for spinless particles and then for particles with spin 1/2.

2 Quantum corrections to metrics

The use of the metric field in General Relativity is inherently classical. However, there is a well defined context that one can discuss quantum corrections to the metric. This occurs when it is quantum matter, not quantum gravity, that is responsible for the quantum effects, and when matter yields such effects which are larger at large distances than the effects of quantum gravity. We will show that the massless nature of the photon implies that these conditions are satisfied for the metrics around charged particles. In these cases the long range propagation of photons provides the quantum modifications of the metric at scales where gravity is still classical.

We will make use of the logic developed in the study of effective field theories. Effective field theory techniques have been developed primarily for situations where there are multiple scales in the problem and are used to identify effects from the physics relevant at the lowest energy scales or largest distance scales. In our case, the highest scale is the photon mass, i.e. zero energy, so that we will be separating the effects of massive degrees of freedom from the massless ones².

²Note that effective field theory techniques are often applied to nonrenormalizable

Gravity couples to the energy momentum tensor $T_{\mu\nu}$ of a particle. We will calculate the energy momentum tensor in a power series in α , using usual Feynman diagram techniques. These calculations are straightforward applications of QED[8]. The result will be expressed in terms of form factors of the various allowed Lorentz structures in the matrix element of $T_{\mu\nu}$. Let us here generically call such a form factor $F(q^2)$, with q_μ being the momentum transfer. Specific cases will be presented later. The form factor is the momentum space description of the structure of the particle. Because the massless photon couples to gravity, this form factor has features not common in most other form factors. Normally, form factors can be expanded in a power series in q^2 around $q^2 = 0$, with the coefficients being related to the structure of the particle. For example, the coefficient of the term linear in q^2 is related to the “charge radius squared” of the particle, i.e.

$$\langle r^2 \rangle = 6 \frac{d}{dq^2} F(q^2) |_{q^2=0} \quad (5)$$

These analytic terms in the power series can equally well be represented by effective Lagrangians with higher powers of derivatives of the fields. However, in the gravitational case, photonic diagrams yield *non-analytic* terms in the expansion of the form factor. In particular, we will find square-root and logarithmic non-analytic terms, i.e.

$$F(q^2) = 1 + a\alpha \frac{q^2}{m^2} \sqrt{\frac{m^2}{-q^2}} + b\alpha \frac{q^2}{m^2} \log(-q^2) + c\alpha \frac{q^2}{m^2} + \dots \quad (6)$$

where a,b,c are constants. These non-analytic terms cannot be represented by effective Lagrangians and can only arise from the long range propagation of massless particles. Note that they imply that the gravitational charge radius is infinite, which reflects the fact that the energy in the electric field extends out to infinity. These non-analytic terms generate the effects that we seek³.

The spatial distribution of energy, and hence the metric, will be recovered by a Fourier transformation to coordinate space. Generically the position

theories. That is not our use here as we are dealing with a renormalizable theory, QED. However, the logic of effective field theory is still useful in this context.

³In chiral perturbation theory these are well known as non-analytic dependences on the pion mass $\sqrt{m_\pi^2}$ and $\log(m_\pi^2)$ [9]. The only other physical situation where there are non-analytic terms in the momentum transfer itself involves massless gravitons in the effective field theory of general relativity[4]

dependent terms in the metric will be

$$\begin{aligned}
\text{metric} &\sim Gm \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{q^2} \left[1 - b\alpha \frac{q^2}{m^2} \sqrt{\frac{m^2}{q^2}} - \frac{q^2}{m^2} \log(q^2) - c\alpha \frac{q^2}{m^2} + \dots \right] \\
&\sim Gm \left[\frac{1}{r} + \frac{a\alpha}{mr^2} + \frac{b\alpha\hbar}{m^2r^3} + \frac{c\alpha}{m^2} \delta^3(x) + \dots \right] \tag{7}
\end{aligned}$$

(Numerical factors of order unity will be inserted later.) The leading piece in the form factor yields the usual “Newtonian” component of the metric. The analytic term in the form factor yields a delta function - i.e. no effect at large distances. (Higher order analytic terms produce derivatives of the delta function.) However, the two non-analytic terms produce the effects that we are interested in. The square-root generates the classical correction in the metric of order α . We will show that this produces precisely the terms required by Einstein’s Equation. The logarithm generates something new which was not present in the classical solution - a term of order $G\alpha\hbar/mr^3$. Here we have reinserted powers of \hbar to emphasize that this is a quantum correction.

It is interesting that a Feynman diagram calculation can generate the *classical* correction in the metric. However, we will demonstrate that this is simply the long-range electromagnetic field which surrounds a charged particle. The logarithm comes from the same class of Feynman diagrams - it encompasses the quantum fluctuations of the long-range field. The fact that it is the *long-range* component that determines the non-analytic behavior indicates that the internal structure of the particle is not relevant. Short range internal structure, for example using a proton instead of an electron, can be represented by a Taylor series in the form factor - and hence only involve analytic terms.

The non-analytic terms are unambiguous finite effects in QED. How can we be sure that other quantum effects are not larger than these? Again the logic of effective field theory allows us to answer this. Quantum effects of massive degrees of freedom are always short ranged at low energy. Hence massive fields yield only analytic terms. The only other relevant massless degrees of freedom are gravitons⁴ The long distance classical and quantum gravity effects are also calculable from the non-analytic components using

⁴ Should one or more of the neutrinos be strictly massless, weak interaction effects could also produce long range modifications to the metric. However, dimensional analysis shows that these are smaller than photonic effects.

effective field theory techniques[4]. Because the gravitational interaction has a dimensionful coupling, Newton's constant G , the power counting is different with the correspondence[4]

$$\frac{G\alpha}{r^2} \rightarrow \frac{G^2 m^2}{r^2} \quad (8)$$

$$\frac{G\alpha\hbar}{mr^3} \rightarrow \frac{G^2 m\hbar}{r^3} \quad (9)$$

Therefore, as long as $Gm^2 < \alpha$, the QED effects will be dominant over quantum gravity effects. Note that describing quantum gravity modifications by a change in the metric is not straightforward, since the long-range propagation of gravity also has a quantum modification. This will be addressed in a future paper[10].

Having identified the non-analytic terms which yield the long-range modifications of the metric, we now turn to the extraction of these effects.

3 Extracting the classical and quantum corrections

Defining the metric tensor via

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (10)$$

where $\eta_{\mu\nu} = (1, -1, -1, -1)_{\text{diag}}$ is the usual Minkowski metric, the interaction Hamiltonian has the form[11]

$$H = \int d^3x \frac{1}{2} T^{\mu\nu}(x) h_{\mu\nu}(x) \quad (11)$$

where $T^{\mu\nu}(x)$ is the energy-momentum tensor. The analog of the Maxwell equation is the (linearized) Einstein equation, which has the form, in harmonic gauge— $\partial^\mu h_{\mu\nu} = \frac{1}{2} \partial_\nu h_\mu^\mu$ —

$$\square h_{\mu\nu} = -16\pi G (T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T) \quad (12)$$

where $T = \eta^{\mu\nu} T_{\mu\nu}$ is the trace. The metric for a nearly static source is then recovered via the Green function in either coordinate or momentum space

$$h_{\mu\nu}(x) = -16\pi G \int d^3y D(x-y) (T_{\mu\nu}(y) - \frac{1}{2} \eta_{\mu\nu} T(y)) \quad (13)$$

$$= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} (T_{\mu\nu}(q) - \frac{1}{2}\eta_{\mu\nu}T(q)) \quad (14)$$

3.1 Spinless particles

For a quantum mechanical system, $T_{\mu\nu}$ is represented by the transition density

$$\langle p_2 | T_{\mu\nu}(x) | p_1 \rangle$$

and the conservation condition $\partial^\mu T_{\mu\nu} = 0$ together with the requirement that $T_{\mu\nu}$ transform as a second rank tensor demands the general (scalar field) form

$$\langle p_2 | T_{\mu\nu}(x) | p_1 \rangle = \frac{e^{i(p_2-p_1)\cdot x}}{\sqrt{4E_2E_1}} \left[2P_\mu P_\nu F_1(q^2) + (q_\mu q_\nu - g_{\mu\nu}q^2) F_2(q^2) \right] \quad (15)$$

where $q_\mu = (p_2 - p_1)_\mu$ is the momentum transfer and $q^2 = q_0^2 - \vec{q}^2$. As can be seen from the condition

$$\langle p_2 | \hat{P}_\mu | p_1 \rangle = \langle p_2 | \int d^3x T_{\mu 0}(x) | p_1 \rangle = P_\mu \langle p_2 | p_1 \rangle \quad (16)$$

conservation of energy-momentum requires $F_1(q^2 = 0) = 1$ but there exists no such constraint on $F_2(q^2)$. In QED at lowest order in α we have $F_1 = 1$ and $F_2 = -1/2$.

In order to calculate these form factors in QED, we calculate the diagrams of Fig. 1. Although we display the result of the full calculation below, the only diagrams relevant for the non-analytic terms are Figs. 1c and 1e, where the graviton couples to the photon. This are the only diagrams with long range propagation from a massless particle. The form factors near $q^2 = 0$ are found to be

$$\begin{aligned} F_1(q^2) &= 1 + \frac{\alpha}{4\pi} \frac{q^2}{m^2} \left(-\frac{8}{3} + \frac{3}{4} \frac{m\pi^2}{\sqrt{-q^2}} + 2 \log \frac{-q^2}{m^2} - \frac{4}{3} \log \frac{\lambda}{m} \right) + \dots \\ F_2(q^2) &= -\frac{1}{2} + \frac{\alpha}{4\pi} \left(-\Omega - \frac{26}{9} + \frac{m\pi^2}{2\sqrt{-q^2}} + \frac{4}{3} \log \frac{-q^2}{m^2} \right) + \dots \end{aligned} \quad (17)$$

where

$$\Omega = \frac{2}{\epsilon} - \gamma - \log \frac{m^2}{4\pi\mu^2}$$

is an ultraviolet divergence, which can be absorbed into the coefficient of a term

$$\mathcal{L} = KR F_{\mu\nu} F^{\mu\nu} \text{tr} QUQU^\dagger. \quad (18)$$

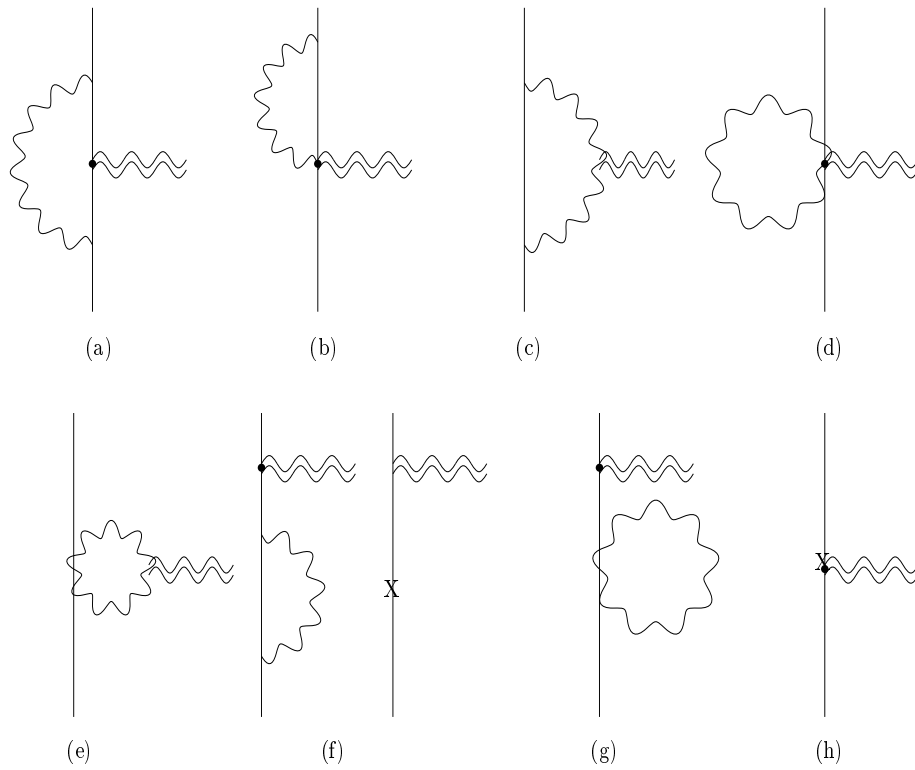


Figure 1: Feynman diagrams for spin 0 radiative corrections to $T_{\mu\nu}$.

in the effective Lagrangian. There are a number of features here which are worthy of note. One is that the $q^2 = 0$ value of the leading form factor $F_1(q^2)$ is unchanged from its lowest order size, as required by energy-momentum conservation, while the form factor $F_2(q^2)$ is modified at $q^2 = 0$. However, the most important new effect is the appearance of nonanalytic terms $\sim \sqrt{-q^2}, q^2 \log -q^2$, etc. in the form factors. The square roots are associated with Fig. 1c only, in which the massive scalar can exist close to its mass shell, while the logarithms come from both Fig. 1c and 1e.

We can explicitly demonstrate that the non-analytic terms are associated with long-range components of the energy momentum tensor by transformation to the coordinate space representation

$$T_{\mu\nu}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} T_{\mu\nu}(\vec{q}) \quad (19)$$

We work in the Breit frame where $q_0 = 0$ and $\vec{p}_2 = -\vec{p}_1 = \vec{q}/2$. The general relationships are

$$\begin{aligned} T_{00}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(mF_1(-\vec{q}^2) + \frac{\vec{q}^2}{2m} F_2(-\vec{q}^2) \right) \\ T_{0i}(\vec{r}) &= 0 \\ T_{ij}(\vec{r}) &= \frac{1}{2m} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (q_i q_j - \delta_{ij} \vec{q}^2) F_2(-\vec{q}^2) \end{aligned} \quad (20)$$

This calculation involves a set of integrals which are listed in the Appendix. Performing the Fourier transform for the one loop form factors we find

$$\begin{aligned} T_{00}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(m - \frac{\alpha\pi|\vec{q}|}{8} - \frac{\alpha\vec{q}^2}{3\pi m} \log \vec{q}^2 \right) + \dots \\ &= m\delta^3(\vec{r}) + \frac{\alpha}{8\pi r^4} - \frac{\alpha\hbar}{\pi^2 m r^5} + \dots \\ T_{0i}(\vec{r}) &= 0 \\ T_{ij}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (q_i q_j - \delta_{ij} \vec{q}^2) \left(\frac{\alpha\pi}{16|\vec{q}|} + \frac{\alpha}{6\pi m} \log \vec{q}^2 \right) + \dots \\ &= -\frac{\alpha}{4\pi r^4} \left(\frac{r_i r_j}{r^2} - \frac{1}{2} \delta_{ij} \right) - \frac{\alpha\hbar}{3\pi^2 m r^5} \delta_{ij} + \dots \end{aligned} \quad (21)$$

Short range modifications of the matter distribution may smear out the delta function but will not change the long range fields. The ‘‘physics’’ origin of

these quantum corrections can be understood in terms of the position uncertainty associated with quantum mechanics which implies the replacement of the distance r in the classical expression by the value $\sim r + \frac{\hbar}{m}$. Since for macroscopic distances $\hbar/m \ll r$ expansion of the classical result in powers of $1/r$ yields to the form of the quantum modifications found in our one loop calculation.

Even though these forms for the energy momentum tensor were calculated from Feynman diagrams, we can verify that what we have called the classical component does in fact represent the classical energy momentum contained in the electric field around a charged particle. Since for electromagnetism[12]

$$T_{\mu\nu}^{EM} = -F_{\mu\lambda}F_{\nu}^{\lambda} + \frac{1}{4}\eta_{\mu\nu}F_{\lambda\delta}F^{\lambda\delta} \quad (22)$$

we expect the energy momentum tensor in the region around a charged mass to be

$$\begin{aligned} T_{00}^{EM}(\vec{r}) &= \frac{1}{2}E^2 = \frac{\alpha}{8\pi r^4} \\ T_{0i}^{EM}(\vec{r}) &= 0 \\ T_{ij}^{EM}(\vec{r}) &= -E_i E_j + \frac{1}{2}\delta_{ij}E^2 = -\frac{\alpha}{4\pi r^4} \left(\frac{r_i r_j}{r^2} - \frac{1}{2}\delta_{ij} \right) \end{aligned} \quad (23)$$

This demonstrates the equivalence of the square-root non-analytic terms in the form factor with the classical field surrounding the particle. The logarithmic terms clearly yield the energy momentum of quantum fluctuations in this field.

From the form factors we may also directly reproduce the metric. Here the metric is determined from the two form factors by

$$\begin{aligned} h_{00}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} F_1(-\vec{q}^2) - \frac{\vec{q}^2}{4m} F_2(-\vec{q}^2) \right) \\ h_{0i}(\vec{r}) &= 0 \\ h_{ij}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} F_1(-\vec{q}^2) \delta_{ij} + \frac{1}{2m} (q_i q_j + \frac{1}{2} \delta_{ij} \vec{q}^2) F_2(-\vec{q}^2) \right) \end{aligned} \quad (24)$$

The leading constant term in the form factor obviously reproduces the usual Newtonian term in the metric. The analytic terms reproduce a Dirac delta

function, or derivatives of delta functions. These have no long range components and we do not display the results. Again the required integrals are collected in the Appendix. The results using the one loop form factors are

$$\begin{aligned}
h_{00}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} - \frac{\alpha\pi|\vec{q}|}{8} - \frac{\alpha\vec{q}^2}{3\pi m} \log \vec{q}^2 \right) + \dots \\
&= -\frac{2Gm}{r} + \frac{G\alpha}{r^2} - \frac{8G\alpha\hbar}{3\pi mr^3} + \dots \\
h_{0i}(\vec{r}) &= 0 \\
h_{ij}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\delta_{ij} \frac{m}{2} + \frac{\alpha\pi}{16|\vec{q}|} (q_i q_j - \delta_{ij} \vec{q}^2) \right. \\
&\quad \left. + \frac{\alpha}{6\pi m} (q_i q_j - \delta_{ij} \vec{q}^2) \log \vec{q}^2 \right) + \dots \\
&= -\delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^4} + \frac{4G\alpha\hbar}{3\pi mr^3} \left(\frac{r_i r_j}{r^2} - \delta_{ij} \right) + \dots
\end{aligned} \tag{25}$$

These are precisely the appropriate forms for the Reissner-Nordström metric—Eq. 1—along with the associated quantum corrections.

3.2 Spin 1/2

Let us now turn our attention to the case of a particle with spin, in particular spin one-half. The general form for the spin 1/2 matrix element of the energy-momentum tensor can be written as[13]

$$\begin{aligned}
\langle p_2 | T_{\mu\nu} | p_1 \rangle &= \bar{u}(p_2) \left[F_1(q^2) P_\mu P_\nu \frac{1}{m} \right. \\
&\quad - F_2(q^2) \left(\frac{i}{4m} \sigma_{\mu\lambda} q^\lambda P_\nu + \frac{i}{4m} \sigma_{\nu\lambda} q^\lambda P_\mu \right) \\
&\quad \left. + F_3(q^2) (q_\mu q_\nu - g_{\mu\nu} q^2) \frac{1}{m} \right] u(p_1)
\end{aligned} \tag{26}$$

The normalization condition $F_1(q^2 = 0) = 1$ corresponds to energy-momentum conservation as found before, while the second normalization condition $F_2(q^2 = 0) = 1$ is required by the constraint of angular momentum conservation. This can be seen by defining

$$\begin{aligned}
\hat{M}_{12} &= \int d^3x (T_{01} x_2 - T_{02} x_1) \\
&\xrightarrow{q \rightarrow 0} -i \nabla_{q_2} \int d^3x e^{i\vec{q}\cdot\vec{r}} T_{01}(\vec{r}) + i \nabla_{q_1} \int d^3x e^{i\vec{q}\cdot\vec{r}} T_{02}(\vec{r})
\end{aligned} \tag{27}$$

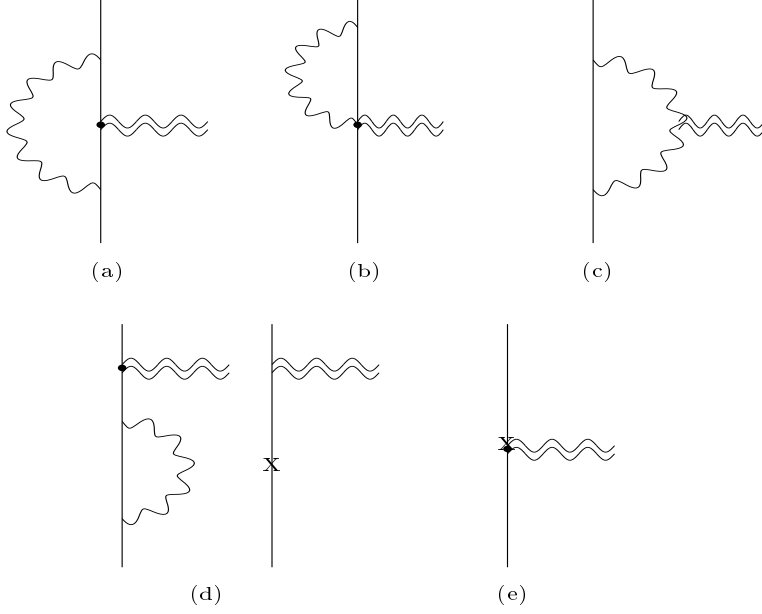


Figure 2: Feynman diagrams for spin 1/2 radiative corrections to $T_{\mu\nu}$.

Then we find

$$\lim_{q \rightarrow 0} \langle p_2 | \hat{M}_{12} | p_1 \rangle = \frac{1}{2} = \frac{1}{2} \bar{u}_\uparrow(p) \sigma_3 u_\uparrow(p) F_2(q^2) \quad (28)$$

i.e., $F_2(q^2 = 0) = 1$, as found explicitly in our calculation.

The Feynman diagrams for fermions are shown in Fig 2. In this case only one diagram, Fig. 2c, will be relevant for the non-analytic terms. We find

$$\begin{aligned} F_1(q^2) &= 1 + \frac{\alpha}{4\pi} \frac{q^2}{m^2} \left(-\frac{39}{18} + \frac{3\pi^2 m}{4\sqrt{-q^2}} + 2 \log \frac{-q^2}{m^2} - \frac{4}{3} \log \frac{\lambda}{m} \right) + \dots \\ F_2(q^2) &= 1 + \frac{\alpha}{4\pi} \frac{q^2}{m^2} \left(-\frac{47}{18} + \frac{1}{2} \frac{\pi^2 m}{\sqrt{-q^2}} + \frac{2}{3} \log \frac{-q^2}{m^2} - \frac{4}{3} \log \frac{\lambda}{m} \right) + \dots \\ F_3(q^2) &= \frac{\alpha}{4\pi} \left(-\frac{11}{18} + \frac{\pi^2 m}{4\sqrt{-q^2}} + \frac{2}{3} \log \frac{-q^2}{m^2} \right) + \dots \end{aligned} \quad (29)$$

We convert this into an energy-momentum tensor. Writing $\vec{S} = \vec{\sigma}/2$ for the spin, the general relation to the fermion form factors is

$$T_{00}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(m F_1(-\vec{q}^2) + \frac{\vec{q}^2}{m} F_3(-\vec{q}^2) \right)$$

$$\begin{aligned}
T_{0i}(\vec{r}) &= i \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{2} (\vec{S} \times \vec{q})_i F_2(-q^2) \\
T_{ij}(\vec{r}) &= \frac{1}{m} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (q_i q_j - \delta_{ij} q^2) F_3(-q^2)
\end{aligned} \tag{30}$$

The calculation may be performed using the integrals listed in the Appendix and we find

$$\begin{aligned}
T_{00}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(m - \frac{\alpha\pi}{8} |\vec{q}| - \frac{\alpha}{3\pi m} q^2 \log q^2 \right) + \dots \\
&= m\delta^3(\vec{r}) + \frac{\alpha}{8\pi r^4} - \frac{\alpha\hbar}{\pi^2 m^2 r^5} + \dots \\
T_{0i}(\vec{r}) &= \frac{i}{2} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (\vec{S} \times \vec{q})_i \left(1 - \frac{\alpha\pi}{8m} |\vec{q}| - \frac{\alpha}{6\pi m^2} q^2 \log q^2 \right) + \dots \\
&= \frac{1}{2} (\vec{S} \times \vec{\nabla})_i \delta^3(\vec{r}) + \left(-\frac{\alpha}{4\pi m r^6} + \frac{5\alpha\hbar}{4\pi^2 m^2 r^7} \right) (\vec{S} \times \vec{r})_i + \dots \\
T_{ij}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(\frac{\alpha\pi}{16|\vec{q}|} + \frac{\alpha}{6\pi m} \log q^2 \right) (q_i q_j - \delta_{ij} q^2) + \dots \\
&= -\frac{\alpha}{4\pi r^4} \left(\frac{r_i r_j}{r^2} - \frac{1}{2} \delta_{ij} \right) - \frac{\alpha\hbar}{3\pi^2 m r^5} \delta_{ij} + \dots
\end{aligned} \tag{31}$$

As expected, the classical fields are the same as for the scalar case. The fact that the spin-independent quantum terms are also the same seems to be reasonable if they represent the quantum fluctuations of the electromagnetic fields. This result is, however, non-trivial in the Feynman diagram calculation as different diagrams are involved. The form of the spin-dependent corrections can also be understood via simple classical arguments. Since the time-space component of the energy-momentum tensor is given by Eq. 22 as

$$T_{0i} = -(\vec{E} \times \vec{B})_i \tag{32}$$

if we combine the electric field from a point charge as before with the magnetic field which arises from a spinning particle with gyromagnetic ratio g

$$\vec{B} = \frac{eg}{2m} \frac{3\hat{r}\vec{S} \cdot \hat{r} - \vec{S}}{4\pi r^3} \tag{33}$$

the associated classical value of T_{0i} is found to be

$$T_{0i} = -\frac{\alpha g}{8\pi m r^6} (\vec{S} \times \vec{r})_i \tag{34}$$

Obviously agreement with Eq. 31 is found if the Dirac value $g = 2$ is used.

Similarly we can obtain the metric components due to this energy-momentum. The relation of the metric to the fermion form factors is

$$\begin{aligned}
h_{00}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} F_1(-\vec{q}^2) - \frac{\vec{q}^2}{2m} F_3(-\vec{q}^2) \right) \\
h_{0i}(\vec{r}) &= -16\pi G i \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \frac{F_2(-\vec{q}^2)}{2} (\vec{S} \times \vec{q})_i \\
h_{ij}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} F_1(-\vec{q}^2) + \frac{1}{m} (q_i q_j + \frac{1}{2} \delta_{ij} \vec{q}^2) F_3(-\vec{q}^2) \right)
\end{aligned} \tag{35}$$

With the form factor calculated above this yields

$$\begin{aligned}
h_{00}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} - \frac{\alpha\pi|\vec{q}|}{8} \frac{\alpha\vec{q}^2}{6\pi m} \log \vec{q}^2 \right) + \dots \\
&= -\frac{2Gm}{r} + \frac{G\alpha}{r^2} - \frac{8G\alpha\hbar}{3\pi m r^3} + \dots \\
h_{0i}(\vec{r}) &= -16\pi G i \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{1}{2} - \frac{\alpha\pi|\vec{q}|}{16m} - \frac{\alpha\vec{q}^2}{12\pi m^2} \log \vec{q}^2 \right) (\vec{S} \times \vec{q})_i + \dots \\
&= \left(\frac{2G}{r^3} - \frac{G\alpha}{m r^4} + \frac{2G\alpha\hbar}{\pi m^2 r^5} \right) (\vec{S} \times \vec{r})_i + \dots \\
h_{ij}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} \left(\frac{m}{2} \delta_{ij} + \frac{\alpha\pi}{16|\vec{q}|} (q_i q_j - \delta_{ij} \vec{q}^2) \right. \\
&\quad \left. + \frac{\alpha}{6\pi m} (q_i q_j - \delta_{ij} \vec{q}^2) \log \vec{q}^2 \right) + \dots \\
&= -\delta_{ij} \frac{2Gm}{r} + \frac{G\alpha r_i r_j}{r^4} + \frac{4G\alpha\hbar}{3\pi m r^3} \left(\frac{r_i r_j}{r^2} - \delta_{ij} \right) + \dots
\end{aligned} \tag{36}$$

We observe that the diagonal components are identical to those found for the spinless case, as expected, and that there exists a nonvanishing non-diagonal term associated with the spin. We have thus reproduced the Kerr-Newman metric in harmonic gauge—Eq. 2—together with the associated quantum corrections.

4 Conclusions

Above we have examined the radiative corrections to the lowest order gravitational coupling of a massive charged particle. We have seen that the form of the q^2 dependence to these form factors has an important difference from the structure of most other form factors. In addition to the usual analytic terms such as q^2/m^2 , *etc.*, the gravitational form factors of charged particles also include nonanalytic components such as $\sqrt{-q^2}$, $q^2 \log -q^2$, *etc.* which are associated with the feature that the graviton can couple to a massless field—in this case the virtual photon. By transforming to co-ordinate space, we demonstrated that these new forms determine long range corrections to the energy-momentum tensor and, via the Einstein equation, to the gravitational field, whose form and strength are fully constrained and determined by the feature that nature is describable in terms of a quantum field theory. Specifically, leading corrections to the Schwarzschild solution of $\mathcal{O}(G\alpha/r^2)$ are found to correspond to well known classical solutions, and quantum corrections of order $\mathcal{O}(G\alpha\hbar/mr^3)$ are determined. These terms appear too small to be measured. However, their presence is an interesting application of effective field theoretic methods in the case of the gravitational interaction.

Going further, we know that the photon is not the only massless field to which the graviton couples—self interaction associated with the nonlinear nature of gravity guarantees that the graviton also couples to itself. Thus there exists an extension of this work to graviton loop diagrams and to higher order components of the gravitational self interaction. Some of these have already been explored in earlier work by one of us[4], but a more general discussion will be the subject of a future communication[10].

5 Appendix

Here we collect the integrals that are needed in the Fourier transformation of the non-analytic terms. For the calculation of the long range part of the energy momentum tensor we use

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} |\vec{q}| = -\frac{1}{\pi^2 r^4} \quad , \quad \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q_j |\vec{q}| = \frac{-4ir_j}{\pi^2 r^6}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{q_i q_j}{|\vec{q}|} = \frac{1}{\pi^2 r^4} \left(\delta_{ij} - 4 \frac{r_i r_j}{r^2} \right) \quad (37)$$

and

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \bar{q}^2 \log \bar{q}^2 = \frac{3}{\pi r^5} \quad , \quad \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q_j \bar{q}^2 \log \bar{q}^2 = \frac{i15r_j}{\pi r^7}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q_i q_j \log \bar{q}^2 = \frac{1}{\pi r^5} \delta_{ij} \quad (38)$$

In calculating the metric we need

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\vec{q}|} = \frac{1}{2\pi^2 r^2} \quad , \quad \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{q_j}{|\vec{q}|} = \frac{ir_j}{\pi^2 r^4}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{q_i q_j}{|\vec{q}|^3} = \frac{1}{2\pi^2 r^2} \left(\delta_{ij} - 2 \frac{r_i r_j}{r^2} \right) \quad (39)$$

for the square-roots and

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \log \bar{q}^2 = -\frac{1}{2\pi r^3} \quad , \quad \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q_j \log \bar{q}^2 = \frac{-i3r_j}{2\pi r^5}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \log \bar{q}^2 \left(\frac{q_i q_j}{\bar{q}^2} \right) = -\frac{r_i r_j}{2\pi r^5} \quad (40)$$

for the logarithms.

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