Lecture 3

6 October 2016

1 Effective field theory

Doing physics, we are usually interested in phenomena at particular energy scales. Given a full theory at hand, one can perform computations at any energy within its range of applicability. Often the computations can be made easier by restricting the theory to some particular range of scales. For example, doing physics at low energies, one may reasonably guess that the influence of high–energy degrees of freedom (DOFs) can be consistently taken into account without the need to directly compute corresponding contributions. In this way we arrive at Effective field theories (EFT). They can be very useful. Indeed, avoiding the complications of a full theory, we can simplify our calculations. What is more important, a full theory may not even be known, e.g. as with gravity, yet the corresponding EFT exists and allows for consistent study of processes at a certain range of energies. Due to the lack of an experimentally verified "theory of everything", all of our real world QFTs are merely EFTs.

Although we do not generally have detailed knowledge about high–energy dynamics when doing low–energy physics this does not mean that this dynamics does not affect EFT. All EFTs are sensitive to high energies to some order. For example, when going to low energies involves spontaneous symmetry breaking (SSB), the symmetric phase of the theory manifests itself in the structure of interactions of a low–energy theory. As a more general example, when one computes loop corrections in EFT, the UV dynamics manifests itself in the running of coupling constants with energy. The effect of heavy DOFs is also typically encoded in operators suppressed by some cutoff scale [1],[2].

1.1 Three principles of effective field theories

What makes us sure that one can overcome the influence of UV scales on low–energy physics? The answer is three–fold. On the one hand, this is locality principle. Speaking loosely, the uncertainty principle,

$$\Delta x \Delta p \sim \hbar, \tag{1}$$

implies that the higher is the energy, the smaller is the distance. Hence one can expect that effects of UV physics are local, and hence these effects can be captured by local operators. As a simple illustration, consider the electron–positron scattering process in QED, $e^+e^- \to e^+e^-$. The tree–level photon propagator behaves as

$$\frac{e_0^2}{q^2}, \tag{2}$$

where e_0 is a bare electron charge and q is a momentum transfer. Summing up 1–particle reducible diagrams leads to the renormalization of the charge,

$$e^2 = \frac{e_0^2}{1 - \Pi(q^2)}. \tag{3}$$

On the other hand, we know that QED is the part of the Standard Model, and the photon propagator gets renormalized by, e.g., a heavy particle. At low energies, $q^2 \ll m_H^2$, the heavy particle contribution to $\Pi(q^2)$ is

$$\Pi(q^2) = \frac{e_0^2}{12\pi^2}\left(\frac{1}{\epsilon} + \ln 4\pi - \gamma - \ln\frac{m_H^2}{\mu^2} + \frac{q^2}{5m_H^2} + ...\right). \tag{4}$$

This is the example of how heavy DOFs participate in the renormalization of the local EFT parameters. Note that the shift in the fine structure constant made by the heavy particle cannot be directly observed since the values of couplings are to be measured experimentally. Had we defined e_{ph} in the limit $q \to 0$, the correction to the propagator would be

$$\frac{1}{q^2}\frac{e^2}{1-\Pi(q^2)} = \frac{e_{ph}^2}{q^2} + \frac{e_0^2}{12\pi^2}\frac{q^2}{5m_H^2}\frac{1}{q^2} + \dots . \tag{5}$$

We see that in the limit $m_H \to \infty$, the UV physics is completely decoupled, and we come back to QED with a modified electron charge. Note that there are some caveats about this decoupling. For example, for a top quark there are many diagrams that do not vanish in the limit $m_t \to \infty$. Instead, they behave as m_t^2 or $\ln(m_t^2)$. This is because the electroweak theory with the $t-$quark removed violates the $SU(2)_L$ symmetry, as the doublet $\begin{pmatrix} t \\ b \end{pmatrix}$ is no longer present. However, it one takes the limit $m_{t,b} \to \infty$ simultaneously, the symmetry is preserved and decoupling occurs.

Let us demonstrate explicitly how the integration out of heavy DOFs leaves us with the local low-energy physics. Consider the theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi\partial_\mu\varphi - m^2\varphi^2) + \varphi F(\psi) + \mathcal{L}(\psi), \tag{6}$$

where the field ψ is assumed to be light compared to φ. Denote

$$Z_0 = \int [d\varphi] e^{i\int d^4x \mathcal{L}(\varphi)}. \tag{7}$$

The partition function of the theory is then written as follows,

$$Z = Z_0^{-1} \int [d\varphi][d\psi] e^{i\int d^4x(\mathcal{L}(\varphi)+\mathcal{L}(\varphi,\psi)+\mathcal{L}(\psi))} =$$

$$Z_0^{-1} \int [d\psi] e^{i\int d^4x \mathcal{L}(\psi))} \int [d\varphi] e^{i\int d^4x(\mathcal{L}(\varphi)+\mathcal{L}(\varphi,\psi))} \equiv Z_0^{-1} Z_1 \int [d\psi] e^{i\int d^4x \mathcal{L}(\psi))} . \tag{8}$$

Integrating by parts, we have

$$\mathcal{L}(\varphi,\psi) + \mathcal{L}(\psi) = -\frac{1}{2}\varphi(\Box + m^2)\varphi + \varphi F(\psi) . \tag{9}$$

Let us now define

$$\tilde{\varphi}(x) = \varphi(x) + \int d^4y D_F(x-y) F(\psi(y)) , \tag{10}$$

where $D_F(x-y)$ is the Green function of the field φ,

$$(\Box + m^2) D_F(x-y) = -\delta^{(4)}(x-y) . \tag{11}$$

Then it follows that

$$-\frac{1}{2}\varphi(\Box+m^2)\varphi+\varphi F(\psi) = -\frac{1}{2}\tilde{\varphi}(\Box+m^2)\tilde{\varphi}-\frac{1}{2}\int d^4y F(\psi(x)) D_F(x-y) F(\psi(y)) . \tag{12}$$

Since $\tilde{\varphi}$ is obtained from φ by a mere shift, the integration measure remains the same, $[d\varphi] = [d\tilde{\varphi}]$. Therefore, we have

$$Z = \int [d\psi] e^{i\int d^4x \mathcal{L}(\psi)} e^{-\frac{i}{2}\langle FDF \rangle} , \tag{13}$$

where we denote, schematically,

$$\langle FDF \rangle = \int d^4x d^4y F(\psi(x)) D_F(x-y) F(\psi(y)) . \tag{14}$$

One clearly sees that the term (14) is non–local in general. This is to be expected since we removed part of the local interactions of the original theory. Note that in deriving (13) no approximation was used, hence the procedure of excluding some fields from the dynamics of the theory is quite general [1]. But in our case we can go further and see that the remaining

[1] In practice, integration out of some DOFs is performed when one is interested only in a part of a content of the original theory. In the example given above we could say that it is dynamics of the field ψ that we wish to study, and treat the field φ as a background to be integrated out. Non–local terms then give rise to dissipation in the reduced theory [3].

theory is actually local. Indeed, consider the propagator

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-y)}}{q^2-m^2} = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \left(-\frac{1}{m^2} - \frac{q^2}{m^4} + ...\right) = \tag{15}$$

$$\left(-\frac{1}{m^2} + \frac{\Box}{m^4} + ...\right) \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} . \tag{16}$$

The last integral is nothing but the delta function $\delta^{(4)}(x-y)$. We arrive at an infinite series of local expressions. Introduce the effective Lagrangian of the theory,

$$Z = \int [d\psi] e^{i\int d^4x \mathcal{L}_{eff}} , \tag{17}$$

then

$$\mathcal{L}_{eff} = \mathcal{L}(\psi) + \frac{1}{2} F(\psi) \frac{1}{m^2} F(\psi) - \frac{1}{2m^4} F(\psi) \Box F(\psi) + ... \tag{18}$$

We observe that as long as $q^2/m^2 \ll 1$, one can restrict ourselves to the finite amount of terms in the expansion of $\langle FDF \rangle$, and hence the effective theory enjoys locality. When $m \to 0$ this property breaks down as the propagator (15) becomes

$$D_F(x-y) \sim \frac{1}{16\pi^2} \frac{1}{(x-y)^2 - i\epsilon} \tag{19}$$

so we see that massless particles cannot be integrated out in the same way that massive ones are.

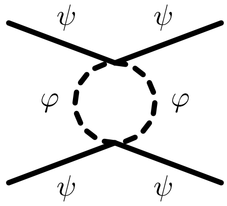
The derivative expansion obtained before is a generic feature of effective field theory. It is the second organizing principle in building any low–energy theory. It claims that there is always terms of growing dimensions in the effective Lagrangian. They are accompanied by the coupling constants which, on dimensional ground, have lowering dimensions. Dimensional analysis allows one to divide the effective Lagrangian into pieces

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \mathcal{L}_{d=5} + \mathcal{L}_{d=6} + ... , \tag{20}$$

where the piece $\mathcal{L}_{d=5}$ contains operators of dimension 5 and so on. The higher dimansional operators is an essential part of the effective field theory. Their presence means that UV physics affects the low–energy behaviour but does this in a controlled way. In fact, one can successfully study low–energy physics without knowing anything about the UV completion of the theory. In this case, all possible higher order operators in (20) represent the effects of unknown UV physics [2].

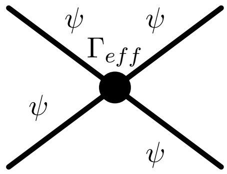
[2] Taking into account the higher order operators is important when one studies the phenomena involving the energies of the order of the UV cutoff of the theory. Perhaps, the most illustrative example is the study of the electroweak vacuum decay, where the answer (the lifetime of the metastable vacuum) can be extremely sensitive to the M_P–suppressed operators [4].

The expression (18) is the form in which the effective Lagrangian is usually used. It represents a valid QFT with the Feynman rules induced from the corresponding UV theory. For example, the diagram



(21)

with the heavy particle running in the loop reduces to the four–vertex diagram



(22)

Let us now make some concluding remarks.

- Higher order operators in the derivative expansion spoil the renormalizability of the theory. Hence, in general EFT is not renormalizable (though without these operators it could have been). Divergences coming from non–renormalizable operators are local.
- Using the locality feature of EFT, the procedure of separating low–energy DOFs from high–energy ones is essential for the effective field theory.
- We have seen that heavy d.o.f. participate in the renormalization of propagators and vertices of EFT resulting in running of coupling constants. If a full theory is unknown, we can use experiment to measure the coefficient of the effective lagrangian. If the theory is known, any predictions of EFT must match those obtained in the framework of the full theory. Perhaps, the most known example of the latter situation is the electroweak theory whose low–energy limit is the Fermi theory. The matching/measuring condition constitutes the third organizing principle of any EFT.

1.2 The linear sigma–model

To illustrate the general considerations made above, we now turn to a particular example – the linear sigma–model. This is one of the most instructive of all field theory models. The full theory is taken to be

$$\mathcal{L}(\sigma,\pi,\psi)=\frac{1}{2}((\partial_\mu\sigma)^2+(\partial_\mu\vec{\pi})^2)+\frac{\mu^2}{2}(\sigma^2+\vec{\pi}^2)-\frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2+ \bar{\psi}i\partial\!\!\!/\psi+g\bar{\psi}(\sigma+i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi\,, \quad (23)$$

where $\vec{\tau}$ are the generators of $SU(2)$ group. The DOFs of the theory are the scalar σ, the triplet of scalars $\vec{\pi}$, and the Dirac fermion ψ. It is useful to quote an alternative form of the theory achieved by redefinition $\Sigma = \sigma + i\vec{\tau}\cdot\vec{\pi}$,

$$\mathcal{L}(\Sigma, \psi) = \frac{1}{4}\mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \frac{\mu^2}{4}\mathrm{Tr}(\Sigma^\dagger \Sigma)\frac{\lambda}{16}(\mathrm{Tr}(\Sigma^\dagger \Sigma))^2 + \\ \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - g(\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma^\dagger \psi_L)\,, \tag{24}$$

where

$$\psi_L = \frac{1}{2}(1+\gamma_5)\psi, \quad \psi_R = \frac{1}{2}(1-\gamma_5)\psi\,. \tag{25}$$

The model is invariant under the global $SU(2)_L \times SU(2)_R$ group. Indeed, if we set

$$\psi_L \to L\psi_L, \quad \psi_R \to R\psi_R, \quad \Sigma \to L\Sigma R^\dagger\,, \tag{26}$$

where $L, R \in SU(2)$, then all combinations of the fields $\psi_{L,R}$ and Σ in (24) are invariant.

Let $\mu^2 > 0$. Then the model allows for SSB. The vacuum solution is

$$\langle\sigma\rangle = \sqrt{\frac{\mu^2}{\lambda}} \equiv v, \quad \langle\vec{\pi}\rangle = 0\,. \tag{27}$$

Consider perturbations above the vacuum parametrized by $\vec{\pi}$ and $\tilde{\sigma} = \sigma - v$. The Lagrangian (23) is rewritten as

$$\mathcal{L} = \frac{1}{2}((\partial_\mu\tilde{\sigma})^2 - 2\mu^2\tilde{\sigma}^2) + \frac{1}{2}(\partial_\mu\vec{\pi})^2 - \lambda v\tilde{\sigma}(\tilde{\sigma}^2 + \vec{\pi}^2) - \tag{28}$$

$$\frac{\lambda}{4}(\tilde{\sigma}^2 + \vec{\pi}^2)^2 + \bar{\psi}(i\partial\!\!\!/ - gv)\psi - g\bar{\psi}(\tilde{\sigma} - i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi\,. \tag{29}$$

The Lagrangian (28) describes the same physics as (23) and enjoys the same $SU(2)_L \times SU(2)_R$ symmetry, though this is not obvious from its form. The symmetry of the unbroken phase manifests itself in the form of interactions of the sigma–model. Observe that the pion fields $\vec{\pi}$ are massless. They are Goldstone fields associated with the broken chiral symmetry.

The Lagrangian (28) is not the only way to represent the low–energy DOFs. For the purposes of constructing the EFT, it is convenient to introduce new fields as follows,

$$U = e^{\frac{i\vec{\tau}\cdot\vec{\pi}'}{v}}, \quad v + \tilde{\sigma} + i\vec{\tau}\cdot\vec{\pi} = (v+s)U\,, \tag{30}$$

where at the linear order $\vec{\pi}' = \vec{\pi} + ...$, and hence $s = \tilde{\sigma} + ...$ We get one more form of the Lagrangian,

$$\mathcal{L} = \frac{1}{2}((\partial_\mu s)^2 - 2\mu^2 s^2) + \frac{(v+s)^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) - \\ \lambda v s^3 - \frac{\lambda}{4}s^4 + \bar{\psi}i\partial\!\!\!/\psi - g(v+s)(\bar{\psi}_L U\psi_R + \bar{\psi}_R U^\dagger\psi_L)\,. \tag{31}$$

This Lagrangian is invariant under $SU(2)_L \times SU(2)_R$ provided that $U \to LUR^\dagger$. We see that the field s is massive with the mass $m_s^2 = 2\mu^2$. We can now use the technique described above to integrate this field out. In consistency with the general form of EFT Lagrangian (20), we have

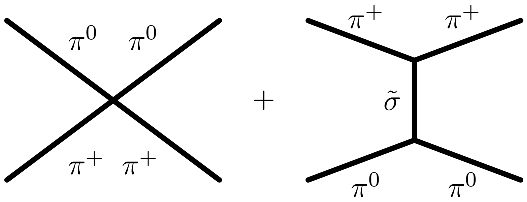
$$\mathcal{L}_{eff} = \frac{v^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{v^2}{8m_s^2}(\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger))^2 + \dots . \tag{32}$$

1.2.1 Test of equivalence

We would like to make sure that all the forms of the UV theory listed above as well as the EFT theory given by (32) give the same result when calculating low–energy processes. To see this, consider the scattering of two pions, $\pi^+\pi^0 \to \pi^+\pi^0$. Consider first the Lagrangian (28). The part of it contributing to the process takes the form

$$\Delta\mathcal{L} = -\frac{\lambda}{4}(\vec{\pi}\cdot\vec{\pi})^2 - \lambda v \tilde{\sigma}\vec{\pi}^2 . \tag{33}$$

There are two diagrams contributing to the process, and the amplitude is given by



(34)

$$= -i\mathcal{M} = -2i\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m_s^2} = \frac{iq^2}{v^2} + O(q^4) .$$

One of the diagrams shows the current–current interaction usual for EFT. Note also that the amplitude of the process depends on the momentum transfer even at the leading order as the constant pieces of two diagrams cancel.

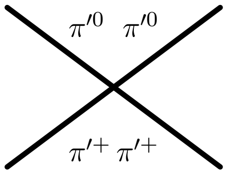
Let us now look at the Lagrangian (31). The part of it relevant for our process takes the form

$$\Delta\mathcal{L} = \frac{(v+s)^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) . \tag{35}$$

Clearly, there is only one four–vertex diagram contributing at the order $O(q^2)$. Expanding (35) to the fourth order in $\vec{\pi}'$, we have

$$\Delta\mathcal{L} = \frac{1}{6v^2}\left[(\vec{\pi}'\cdot\partial_\mu\vec{\pi}')^2 - \vec{\pi}'^2(\partial_\mu\vec{\pi}'\cdot\partial^\mu\vec{\pi}')\right] . \tag{36}$$

The amplitude is given by



$$= -i\mathcal{M} = \frac{iq^2}{v^2} + O(q^4)\,. \tag{37}$$

Finally we look at the EFT Lagrangian (32). One sees that the leading order term contributing to the scattering process coincides with that of (35), hence the amplitude is the same. Here we see the advantage of using EFT approach: it allows us to rewrite the theory in the form at which only relevant at low energies DOFs are present in the Lagrangian. By no means, this simplifies significantly calculations.

The lesson we have learnt from this equivalence test is that the physically measured quantities (like S–matrix elements) should not depend on the choice of variables we use to enumerate DOFs of the theory. This is essentially the statement of the Haag's theorem [5],[6]. Specifically, let the original Lagrangian be $\mathcal{L}(\varphi)$, and let the redefinition of the fields is

$$\varphi = \chi F(\chi), \quad F(0) = 1\,. \tag{38}$$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\chi F(\chi)) \equiv \tilde{\mathcal{L}}(\chi)$. The claim now is that the Lagrangians $\mathcal{L}(\varphi)$ and $\tilde{\mathcal{L}}(\varphi)$ describe the same physic in the sense that on–shell matrix elements computed with either Lagrangian are identical. A little contemplation shows that this is to be expected. Indeed, since $F(0) = 1$, the free theories clearly coincide. But then asymptotic conditions for any scattering experiment written in both theories coincide as well. In turn, as soon as the initial conditions are specified, the result of the experiment cannot depend on which quantities we use to compute the interactions taking place in the middle. To put it in other words, "names do not matter".

The EFT approach outlined above allows to recover all pion physics at low energies. In this sense, the EFT (32) is a full QFT. It can be continued beyond the low orders in q^2 by including terms of higher powers. As is written in (32), it provides us with the correct amplitude for $\pi^+\pi^0 \to \pi^+\pi^0$ scattering process up to $O(q^4)$. The first part gives rise to the four–vertex diagram that contributes at the order of q^2, and the second part leads to the diagram like the rightmost one in Eq. (34), which contributes at the order q^4.

Let us finally quote the partition function of the theory,

$$Z[J] = \int [ds][d\vec{\pi}] e^{i\int d^4x (L_{full}(s,\vec{\pi}) + \vec{J}\cdot\vec{\pi})} = \int [d\vec{\pi}] e^{i\int d^4x (L_{eff}(\vec{\pi} + \vec{J}\cdot\vec{\pi})}\,. \tag{39}$$

From this expression one can derive all the correlation functions, Feynman rules, etc. of the low–energy theory. This again illustrates the fact that the EFT is a viable QFT.

2 Loops

Now let us tackle loop effects within the EFT. Here are the essential points in performing this program:

- the linear σ–model is a renormalizable theory. Thus, one can just compute everything in this theory, renormalize and look at the low–energy limit.
- Instead, one can use an EFT, but this is a non–renormalizable theory. Would it stop us? No, because we can still take loops, renormalize them, and obtain "finite" predictions at low energies.
- Recall that an EFT contains a bunch of unknown parameters. Having computed the loops both in the EFT and in the full theory we can just match the relevant expressions for amplitudes and retrieve the EFT parameters. This procedure is called "matching".

Why does this work? By construction an EFT is not reliable at high energies, but since its effect is local (thanks to the uncertainty principle), it is encoded by local terms in the effective Lagrangian. The low–energy predictions then must be the same in the EFT and the full theory, and thus the EFT is predictive at low energies.

Let us write down the most general EFT Lagrangian up to the next to the leading order in the energy expansion that requires the symmetry under $SU(2)_L \times SU(2)_R$ group,

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) + l_1[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right)]^2 + l_2[\mathrm{Tr}\left(\partial_\mu U \partial_\nu U^+\right)]^2 \,. \quad (40)$$

The invariance is achieved if $U \to LUR^+$, where $L, R \in SU(2)$. Now we apply the background field method and factorize the "background" and "quantum" fields,

$$U = \bar{U} e^{i\Delta}\,, \quad \text{where} \quad \Delta \equiv \vec{\tau}\cdot\vec{\Delta}\,. \quad (41)$$

Then we expand our Lagrangian in Δ, e.g.,

$$\begin{aligned}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) =& \mathrm{Tr}\left(\partial_\mu \bar{U} \partial^\mu \bar{U}^+\right) - 2i\mathrm{Tr}\left(\bar{U}^+ \partial_\mu \bar{U} \partial^\mu \Delta\right) \\ &+ \mathrm{Tr}\left[\partial_\mu \Delta \partial^\mu \Delta + \bar{U}^+ \partial_\mu \bar{U}(\Delta \partial^\mu \Delta - \partial^\mu \Delta \Delta)\right]\,.\end{aligned} \quad (42)$$

The renormalized quadratic action then takes the form

$$S_2^{(0)} = \int d^4x \left\{ \mathcal{L}_2(\bar{U}) - \frac{v^2}{2}\Delta_a (d_\mu d^\mu + \sigma)^{ab}\Delta_b + ... \right\}\,, \quad (43)$$

where

$$
\begin{aligned}
d_\mu^{ab} &= \delta^{ab}\partial_\mu + \Gamma_\mu^{ab} \, , \\
\Gamma_\mu^{ab} &= -\frac{1}{4}\mathrm{Tr}\left([\tau^a, -\tau^b](\bar{U}^+\partial_\mu \bar{U})\right) \, , \\
\sigma^{ab} &= \frac{1}{8}\mathrm{Tr}\left([\tau^a, \bar{U}^+\partial_\mu \bar{U}][\tau^b, \bar{U}^+\partial^\mu \bar{U}]\right) \, .
\end{aligned}
\tag{44}
$$

It is also instructive to recall the heat kernel method, which yields the following diverging part of the 1–loop effective action,

$$
\begin{aligned}
W_{1-loop} =& \frac{i}{2}\mathrm{Tr}\ln(d_\mu d^\mu + \sigma) \\
=& \frac{1}{(4\pi)^{d/2}}\int d^4x \lim_{m\to 0}\left\{\Gamma\left(1-\frac{d}{2}\right) m^{d-2}\mathrm{Tr}\sigma \right. \\
& \left. + m^{d-4}\Gamma\left(2-\frac{d}{2}\right)\mathrm{Tr}\left(\frac{1}{12}\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2\right) + \ldots\right\} ,
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
\mathrm{Tr}\Gamma_{\mu\nu}\Gamma^{\mu\nu} =& \frac{N_f}{8}\mathrm{Tr}\left(\left[\bar{U}^+ D_\mu \bar{U}, \bar{U}^+ D_\nu \bar{U}\right]\left[\bar{U}^+ D^\mu \bar{U}, \bar{U}^+ D^\nu \bar{U}\right]\right) , \\
\mathrm{Tr}\sigma^2 =& \frac{1}{8}\left[\mathrm{Tr}\left(D_\mu \bar{U} D^\mu \bar{U}^+\right)\right]^2 + \frac{1}{4}\mathrm{Tr}\left(D_\mu \bar{U} D_\nu \bar{U}^+\right)\mathrm{Tr}\left(D^\mu \bar{U} D^\nu \bar{U}^+\right) \\
& + \frac{N_f}{8}\mathrm{Tr}(D_\mu \bar{U} D^\mu \bar{U}^+ D_\nu \bar{U} D^\nu \bar{U}^+) \, .
\end{aligned}
\tag{46}
$$

Now we can absorb the divergences into the "renormalized" coupling constants of the theory, which yields

$$
\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) + l_1^r[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right)]^2 + l_2^r[\mathrm{Tr}\left(\partial_\mu U \partial_\nu U^+\right)]^2 \, , \tag{47}
$$

with

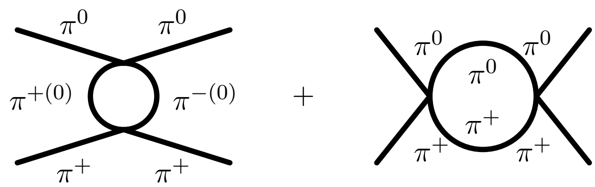
$$
\begin{aligned}
l_1^r &= l_1 + \frac{1}{384\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right] \, , \\
l_2^r &= l_1 + \frac{1}{192\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right] \, .
\end{aligned}
\tag{48}
$$

Now let us study the "finite", non–local contributions. To this end we use the background field method, which gives (see Lecture 2),

$$
\begin{aligned}
&\Delta S_{\mathrm{finite}} = \int d^4x d^4y \; \mathrm{Tr}\left\{\frac{\Gamma_{\mu\nu} L(x-y)\Gamma^{\mu\nu}}{12} + \frac{\sigma(x) L(x-y)\sigma(y)}{2}\right\} , \\
&\text{where} \quad L(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{iq(x-y)} \ln\left(\frac{q^2}{\mu^2}\right) \, .
\end{aligned}
\tag{49}
$$

The 1−loop effective action includes all processes up to $\sim O(\pi^6)$.

Now we can easily compute the amplitude of the pion scattering $\pi^0\pi^+ \to \pi^0\pi^+$ at one loop. In the EFT this amounts to computing only the bubble diagrams,



$$= \mathcal{M}_{eff} = \frac{t}{v^2} + \left[8l_1^r + 2l_2^r + \frac{5}{192\pi^2}\right]\frac{t^2}{v^4} + \left[2l_2^r + \frac{7}{576\pi^2}\right](s(s-u) + u(u-s))/v^4 \\ - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{\mu^2} + s(s-u)\ln\frac{-s}{\mu^2} + u(u-s)\ln\frac{-u}{\mu^2}\right]. \tag{50}$$

At the same time, the $\pi^0\pi^+ \to \pi^0\pi^+$ scattering can be computed in the full sigma–model. In this case the calculation is rather lengthy and one has to take into account the bubble, triangle and box diagrams. The latter have a particularly difficult form, which can be found in [7]. The low–energy limit of the amplitude obtained from the full theory gives,

$$\mathcal{M}_{full} = \frac{t}{v^2} + \left[\frac{v^2}{m_\sigma^2} - \frac{11}{96\pi^2}\right]\frac{t^2}{v^4} - \frac{1}{144\pi^2 v^4}(s(s-u) + u(u-s)) \\ - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right]. \tag{51}$$

Requiring the two expressions, Eq. (51) and Eq. (50), to coincide, we obtain the EFT parameters,

$$l_1^r = \frac{v^2}{8m_\sigma^2} + \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right], \\ l_2^r = \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right]. \tag{52}$$

One can compare this result with the tree–level matching Eq.(32) and conclude that we have taken into account an important kinematic feature – the logarithmic dependence of the coupling constant upon the characteristic momentum transfer in the problem.

We saw that the predictions of the EFT, upon matching, accurately reproduce the results of the full theory. Once matching is done, one can use the EFT to calculate other processes without the need to rematch the couplings again. The effect of the massive particles has been reduced to just a few numbers in the effective Lagrangian, and all low–energy processes are described by the light DOFs. In principle, if the high–energy theory is not known, the EFT couplings can be obtained from measurements.

We have also observed another very important property of the EFT. Naively, one might estimate that loops can contribute at order $O(E^2)$ because loop propagators contain powers of energy in their denominators. However, as we have seen, this is not the case. We have seen that the tree–level amplitude of the $\pi^0\pi^+ \to \pi^0\pi^+$ scattering scales as,

$$\mathcal{M}^{tree}_{\pi^0\pi^+\to\pi^0\pi^+} \sim \frac{q^2}{v^2}\,, \tag{53}$$

while the 1−loop result is,

$$\mathcal{M}^{1-loop}_{\pi^0\pi^+\to\pi^0\pi^+} \sim \frac{q^4}{v^4}\,. \tag{54}$$

Since the external momenta are small, the loop expansion is converging. This happens because every vertex contains a factor $1/v^2$ and thus must be accompanied by momenta squared in the numerator in order to end up in a dimensionless quantity. Thus, the higher are the loops we are going to, the bigger is the overall momentum power of the amplitude.

This statement is known as the Weinberg's power counting theorem. It says, essentially, that the overall energy dimension of a diagram with N_L loops is,

$$D = 2 + \sum_n N_n(n-2) + 2N_L\,, \tag{55}$$

where N_n stands for the number of vertices arising from the subset of effective Lagrangians that contain n derivatives. This gives very simple power–counting rules:

- at order $O(E^2)$ one has to take into account only two–derivative Lagrangians at tree level.
- at order $O(E^4)$ one takes one–loop diagrams made of the $O(E^2)$–terms and the $O(E^4)$ Lagrangians at tree level. Then one renormalizes the $O(E^4)$ Lagrangian.
- at order $O(E^6)$ one takes two–loop diagrams made of the $O(E^2)$–terms, one–loop diagrams made of $O(E^4)$ and $O(E^2)$ terms, and tree–level diagrams coming from the $O(E^6)$ Lagrangian.
- in this way one proceeds to a desired accuracy level.

Before closing this section, let us discuss the regime of validity for an effective field theory. As we have seen, the scattering amplitude scales as, schematically,

$$\mathcal{M} \sim \frac{q^2}{v^2}\left(1 + \frac{q^2}{m_\sigma^2} + ...\right) . \qquad (56)$$

This suggests that the EFT expansion breaks down at energies comparable to the masses of the heavy particles. Thus, the EFT itself reveals its limits. The scale at which the energy expansion breaks down is called "cut–off". In most of situations the EFT cut–off is set by heavy particles' masses, but there exist more subtle examples. For instance, one can integrate out hard modes of some field but keep the low–energy modes of this field as active DOFs in the EFT. This is done, for instance, in the effective Hamiltonian of the weak decays and in the dynamics of turbulent flows.

3 Chiral Perturbation Theory

In this section we will give a brief overview of the Chiral Perturbation Theory (ChPT) which gives the easiest and yet powerful example of an EFT description of the Standard Model at lowest possible energies. The main difference of the ChPT effective Lagrangian with respect to the sigma–model is that the chiral symmetry is to be broken. The QCD Lagrangian reads

$$\mathcal{L}_{QCD} = \sum_{quarks} \left(\bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R - \bar{\psi}_L m \psi_R - \bar{\psi}_R m \psi_L\right) . \qquad (57)$$

If the quarks were massless, QCD would be invariant under the $SU(2)$ chiral transformations,

$$\psi_{L,R} \to (L,R)\psi_{L,R} = \exp\{-i\theta^a_{L,R}\tau_a\}\psi_{L,R} . \qquad (58)$$

The axial symmetry is broken dynamically by the quark condensate, and pions are the corresponding Goldstone bosons (approximately, since they do have masses). The vectorial isospin symmetry remains approximately intact [3], i.e.,

$$SU_L(2) \times SU_R(2) \to SU_V(2) , \qquad (59)$$

which manifests itself in the near equality of the masses of $(\pi^\pm, \pi^0)$, (p, n), etc.

It is clear that in the absence of the pion masses their Lagrangian should take the form,

$$\mathcal{L} = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U\right) , \qquad (60)$$

[3] The vectorial isospin symmetry is broken because $m_u \neq m_d$. The difference $|m_d - m_u| \sim 3$ MeV, however, is much smaller than $\Lambda_{QCD} \sim 250$ MeV, which is why ChPT is isospin symmetric to a very good accuracy.

with

$$U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{F}}\,. \tag{61}$$

Now we have to include the mass term. The way to do this is to introduce a "compensator" field χ which will restore the axial symmetry at the level of the Lagrangian, but then break it spontaneously by acquiring a vacuum expectation value. We consider a free QCD–like Lagrangian coupled to a background complex scalar field $\phi = s + ip$:

$$\mathcal{L} = \bar{\psi}_L \not{D}\psi_L + \bar{\psi}_R \not{D}\psi_R - \bar{\psi}_L(s+ip)\psi_R - \bar{\psi}_R(s-ip)\psi_L\,. \tag{62}$$

The limit $p \to m$, $s \to 0$ reduces this theory to the QCD with the broken chiral symmetry. But in general, one can make this Lagrangian chiral invariant by assuming that ϕ transforms as

$$s + ip \to L(s+ip)R\,. \tag{63}$$

Upon introducing the field χ,

$$\chi \equiv 2B_0(s+ip)\,, \tag{64}$$

with B_0 =const, the low–energy effective Lagrangian for pions can be rewritten as,

$$\mathcal{L}_{eff,\pi} = \frac{F^2}{4}\mathrm{Tr}\;(\partial_\mu U \partial^\mu U) + \frac{F^2}{4}\mathrm{Tr}\;\left(\chi^+ U + U^+\chi\right)\,. \tag{65}$$

At the lowest order we obtain,

$$\mathcal{L}_{eff,\pi} = \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - B_0 s\;\vec{\pi}\cdot\vec{\pi} + F^2 B_0 s\,. \tag{66}$$

The pion mass is generated by the condensate of the u and d quarks. In order for the field χ to reproduce the quark masses one has to break the axial symmetry. To this end one assigns the expectation value of the s field to

$$\begin{aligned} s &= m_u + m_d\,,\\ p &= 0\,, \end{aligned} \tag{67}$$

which reproduces the quark masses and gives the following pion mass:

$$m_\pi^2 = B_0(m_u + m_d)\,. \tag{68}$$

Taking the vacuum expectation value of the u and d quarks' Hamiltonian and that of the chiral theory, we obtain,

$$\langle 0|\bar{\psi}\psi|0\rangle = -\langle 0|\frac{\delta\mathcal{L}_{u,d}}{\delta s}|0\rangle = -\langle 0|\frac{\delta\mathcal{L}_{eff,\pi}}{\delta s}|0\rangle = -F^2 B_0\,. \tag{69}$$

The full EFT program can (and have been) carried out (see Ref. [8] for detail). In this way one should write down all possible operators involving U and χ that are consistent with the chiral symmetry and act along the lines above. In fact, ChPT has been widely used to give predictions for different processes up to 2 loops. The reader is advised to consult Ref. [9] for further details. ChPT thus represents a very successful and predictive framework within which the EFT ideas work at their best.

4 Conclusion

Let us summarize main principles of the EFT approach:

- identify low–energy DOFs and symmetries
- write the most general effective Lagrangian
- order it in the local energy expansion
- calculate starting with the lowest order
- renormalize
- match or measure free parameters of the EFT
- use the EFT to predict residual low–energy effects

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