Lecture 2

29 September 2016

1 The second quantization of weak gravitational field

1.1 The second quantization

In this section we assume the gravitational field to be weak and apply the following ansatz,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

In (1), the field $h_{\mu\nu}$ is to be quantized. Note that the decomposition (1) is not uniqe due to GCT covariance of the theory [1]. To make it unique, one should fix the gauge. The convenient choice is the harmonic gauge which is given by

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0\,. \tag{2}$$

Note that the expression (2) is exact in $h_{\mu\nu}$ and it reduces to

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h^\lambda_\lambda = 0 \tag{3}$$

in the weak field limit. Let us now expand Einstein equations in powers of $h_{\mu\nu}$. Let the matter EOM be $T_{\mu\nu}$, then

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{4}$$

Denote by $G^{(i)}_{\mu\nu}$ the part of $G_{\mu\nu}$ containing the i's power of $h_{\mu\nu}$, then up to the second order

$$G_{\mu\nu} \approx G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu}. \tag{5}$$

Define the tensor $t_{\mu\nu}$ as follows

$$t_{\mu\nu} = -\frac{1}{8\pi G}G^{(2)}_{\mu\nu}. \tag{6}$$

[1] To be more precise, Eq.(1) is covariant with respect to those GCT that preserve the condition $|h_{\mu\nu}| \ll 1$. Later on, speaking about tensorial quantities like $h_{\mu\nu}$ or $t_{\mu\nu}$, we will imply that GCT are restricted to the transformations that keep them small.

Substituting (6) and (5) into (4) gives

$$\Box h_{\mu\nu} \approx 8\pi G(T_{\mu\nu} + t_{\mu\nu}), \tag{7}$$

where we have used $G^{(1)}_{\mu\nu} = \Box h_{\mu\nu}$. Hence higher–order powers of $h_{\mu\nu}$ serve as a source of $h_{\mu\nu}$ itself, in complete agreement with the EP. The tensor $t_{\mu\nu}$ provides us with the triple graviton vertex, and $T_{\mu\nu}$ represents the tree graviton correction to the matter propagator. For completeness, we quote the explicit expression for $t_{\mu\nu}$,

$$\begin{aligned}
t_{\mu\nu} = &-\frac{1}{4}h_{\alpha\beta}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{8}h\partial_\mu\partial_n uh \\
&+\frac{1}{8}\eta_{\mu\nu}\left(h^{\alpha\beta}\Box h_{\alpha\beta} - \frac{1}{2}h\Box h\right) \\
&-\frac{1}{4}\left(h_{\mu\rho}\Box h^\rho\nu + h_{\nu\rho}\Box h^\rho\mu - h_{\mu\nu}\Box h\right) \\
&+\frac{1}{8}\partial_\mu\partial_\nu\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right) - \frac{1}{16}\eta_{\mu\nu}\Box\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right) \\
&-\frac{1}{4}\partial_\alpha\left[\partial_\nu\left(h_{\mu\beta}h^{\alpha\beta}\right) + \partial_\mu\left(h_{\nu\beta}h^{\alpha\beta}\right)\right] \\
&+\frac{1}{2}\partial_\alpha\left[h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta})\right],
\end{aligned} \tag{8}$$

where $h \equiv h^\mu_\mu$. In this expression, the last three lines are actually a total derivative, while the second and the third lines vanish on–shell.

Let us now implement the second quantization procedure to the field $h_{\mu\nu}$. To this end, one should write down the general solution to the linearized equation of motion in the absence of matter. Two possible polarizations of the graviton are captured by introducing the polarization tensor $\epsilon_{\mu\nu}$. The latter can be composed from the usual polarization vectors,

$$\epsilon_\mu(\lambda) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad \lambda = \pm. \tag{9}$$

These vectors satisfy the following relations,

$$\epsilon^*_\mu(\lambda)\epsilon^\mu(\lambda) = -1, \quad \epsilon_\mu(\lambda)\epsilon^\mu(\lambda) = 0. \tag{10}$$

We can now form the polarization tensor,

$$\epsilon_{\mu\nu}(\lambda_1\lambda_2) = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2). \tag{11}$$

The plane–wave decomposition of $h_{\mu\nu}$ is then written as

$$h_{\mu\nu} = \sum_{\lambda=++,--} \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a(p,\lambda)\epsilon_{\mu\nu}(p,\lambda)e^{-ipx} + h.c.\right]. \tag{12}$$

From here, the canonical Hamiltonian of the gravitational field can be readily derived,

$$H=\int d^3x t_{00}=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\omega_p\left[a^{\dagger}(p,\lambda)a(p,\lambda)+\frac{1}{2}\right]. \tag{13}$$

To treat $h_{\mu\nu}$ as a quantum field, we promote the coefficients $a(p,\lambda)$ and $a^{\dagger}(p,\lambda)$ to operators with the canonical commutation relations

$$[a(p,\lambda),a^{\dagger}(p',\lambda')]=\delta(p-p')\delta_{\lambda\lambda'}. \tag{14}$$

1.2 Propagator

We start by expanding the action $S_{EH}+S_m$ to the second order in $h_{\mu\nu}$. It is convenient to introduce the quantity

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h. \tag{15}$$

The Lagrangian is given by

$$\sqrt{-g}\mathcal{L}=\sqrt{-g}\left(-\frac{2}{\kappa^2}R+\mathcal{L}_m+\mathcal{L}_{GF}\right). \tag{16}$$

Here $\mathcal{L}_{GF}$ is the gauge–fixing part of the Lagrangian. To the second order in $h_{\mu\nu}$,

$$-\sqrt{-g}\frac{2}{\kappa^2}R=-\frac{2}{\kappa^2}(\partial_\mu\partial_\nu h^{\mu\nu}-\Box h)+\frac{1}{2}\left[\partial_\lambda h_{\mu\nu}\partial^\lambda\bar{h}^{\mu\nu}-2\partial^\lambda\bar{h}_{\mu\lambda}\partial_\sigma\bar{h}^{\mu\sigma}\right], \tag{17}$$

$$\mathcal{L}_{GF}=\xi\partial_\mu\bar{h}^{\mu\nu}\partial^\lambda\bar{h}_{\lambda\nu}. \tag{18}$$

The harmonic gauge corresponds to $\xi=1$, and in this case the Lagrangian (16) rewrites as

$$\sqrt{-g}\mathcal{L}=\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}-\frac{1}{4}\partial_\lambda h\partial^\lambda h-\frac{\kappa}{2}h^{\mu\nu}T_{\mu\nu}. \tag{19}$$

Integration by parts yields

$$\mathcal{L}=\frac{1}{2}h_{\mu\nu}\Box\left(I^{\mu\nu\alpha\beta}-\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta}-\frac{\kappa}{2}h^{\mu\nu}T_{\mu\nu}, \tag{20}$$

where

$$I_{\mu\nu\alpha\beta}=\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta}). \tag{21}$$

From (20) it is easy to extract the propagator for the gravitational field $h_{\mu\nu}$. The latter is given by the inverse of the expression under box, i.e.

$$\left(I^{\mu\nu\alpha\beta}-\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)D_{\alpha\beta\gamma\delta}=I^{\mu\nu}{}_{\gamma\delta}. \tag{22}$$

Trying the ansatz $D_{\alpha\beta\gamma\delta} = aI_{\alpha\beta\gamma\delta} + b\eta_{\alpha\beta}\eta_{\gamma\delta}$ yields $a = 1$, $b = -\frac{1}{2}$. Hence the propagator in the x–representation is given by

$$iD^{\alpha\beta\gamma\delta}(x) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 + i\epsilon} e^{-iqx} P^{\alpha\beta\gamma\delta}, \tag{23}$$

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right]. \tag{24}$$

1.3 Feynman rules

Now we have all necessary ingredients for deriving Feynman rules for graviton.

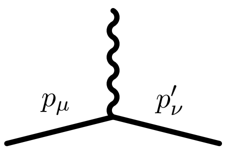
- The propagator reads

$$= \frac{iP^{\alpha\beta\gamma\delta}}{q^2}. \tag{25}$$

- The vertex including the matter propagator can be extracted from the expression $\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}$. Consider, for example, the massive scalar field φ whose EMT is given by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}(\partial_\lambda\varphi\partial^\lambda\varphi - m^2\varphi^2). \tag{26}$$

Then, the corresponding vertex is



$$= i\frac{\kappa}{2}\left[(p_\mu p'_\nu + p'_\mu p_\nu) - \eta_{\mu\nu}(p\cdot p' - m^2)\right]. \tag{27}$$

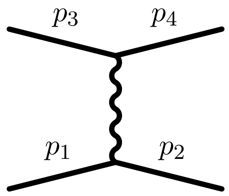
- Much more complicated structure is revealed in the triple graviton vertex,

(28)

$$= i\kappa P_{\alpha\beta,\gamma\delta}\left[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right] + ...$$

We do not quote the full result here, it can be found, e.g., in Ref. [1].

As an example of application of Feynman rules, let us compute the scattering of two scalar particles by a single graviton exchange. The amplitude of the process is given by



$$= -i\mathcal{M}$$

$$= \frac{i\kappa}{2}\left[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu}(p_1 \cdot p_2 - m^2)\right]\frac{i}{q^2}P_{\mu\nu\alpha\beta}\frac{i\kappa}{2}\left[p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - \eta^{\mu\nu}(p_3 \cdot p_4 - m^2)\right]. \tag{29}$$

Consider the non–relativistic limit in which $p^\mu \approx (m, \vec{0})$. The amplitude becomes

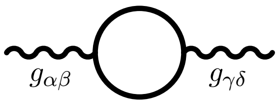
$$\mathcal{M} = -\frac{\kappa^2}{4}\frac{m_1^2 m_2^2}{q^2} = -16\pi G\frac{m_1^2 m_2^2}{q^2} \tag{30}$$

Fourier–transforming the last expression, we obtain the non–relativistic potential

$$V(r) = -\frac{Gm_1 m_2}{r}, \tag{31}$$

which is nothing but Newtonian potential. This completes building GR as QFT at tree level.

What about loop diagrams? Consider, for example, a one–loop matter correction to the graviton propagator. It is given by

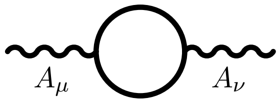


$$= \int \frac{d^4 l}{(2\pi)^2}\frac{i\kappa}{2}\left[l_\alpha(l+q)_\beta + l_\beta(l+q)_\alpha\right]\frac{i}{l^2}\frac{i}{(l+q)^2}\frac{i\kappa}{2}\left[l_\delta(l+q)_\gamma + l_\gamma(l+q)_\delta\right]. \tag{32}$$

Computing this loop, we arrive at the expression of the form, schematically,

$$\frac{\kappa^2}{16\pi^2}(q_\gamma q_\delta q_\alpha q_\beta)\left(\frac{1}{\epsilon} + \ln\ q^2\right). \tag{33}$$

Note the qualitative difference of this result with that of QED,



$$= \frac{e^2}{16\pi^2}(q_\mu q_\nu - \eta_{\mu\nu} q^2)\left(\frac{1}{\epsilon} + \ln q^2\right) . \qquad (34)$$

The divergence in the last expression can be renormalized by the term of the form $\frac{1}{\epsilon} F_{\mu\nu}F^{\mu\nu}$. This is to be expected, since QED is the renormalizable theory. On the contrary, the expression (33) needs terms with 4 derivatives of $h_{\mu\nu}$ to be canceled, and there are no such terms in the Einstein-Hilbert action.

2 Background field method

A particularly powerful way of computing loop corrections in gauge field theories is the background field method.

2.1 Preliminaries

2.1.1 Toy example: scalar QED

Let us start with a pedagogical example of quantum electrodynamics with a massless scalar, described by the "bare" Lagrangian

$$\mathcal{L} = D_\mu \phi (D^\mu \phi)^* - \frac{1}{4} F_{\mu\nu}^2 , \qquad (35)$$

where D_μ stands for the covariant derivative defined as,

$$D_\mu = \partial_\mu + ieA_\mu , \qquad (36)$$

and $F_{\mu\nu}$ denotes the strength tensor of a background electromagnetic field A_μ. Upon integration by parts the Lagrangian (35) can be rewritten as,

$$\begin{aligned} \mathcal{L} = &- \phi(\Box + 2ieA^\mu \partial_\mu + ie(\partial_\mu A^\mu) - e^2 A_\mu^2)\phi * - \frac{1}{4} F_{\mu\nu}^2 , \\ &\equiv -\phi(\Box + v(x))\phi * - \frac{1}{4} F_{\mu\nu}^2 . \end{aligned} \qquad (37)$$

We proceed by performing functional integration over the field ϕ treating the potential $v \sim e, e^2 \ll 1$ as a small perturbation. The overall partition function reads,

$$Z = N_0^{-1} \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}A_\mu \exp\left\{ -i \int d^4x \phi(\Box + v(x))\phi^* - \frac{i}{4} \int d^4x \ F_{\mu\nu}^2 . \right\} \qquad (38)$$

Let us focus on the part with the scalar field,

$$\begin{aligned} &\mathcal{N}^{-1}\int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{-i\int d^4x\phi(\Box+v(x))\phi^*\right\} = \frac{\mathcal{N}^{-1}}{\det(\Box+v(x))} \\ &= \mathcal{N}^{-1}\exp\left\{-\int d^4x\langle x|\mathrm{Tr}\ln(\Box+v(x))|x\rangle\right\}\,, \end{aligned} \tag{39}$$

where the normalization factor is given by,

$$\mathcal{N}\equiv\int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{-i\int d^4x\phi\Box\phi^*\right\} = \exp\left\{-\int d^4x\langle x|\mathrm{Tr}\ln\Box|x\rangle\right\}. \tag{40}$$

We evaluate the operator $\mathrm{Tr}\ln(\Box+v(x))$ perturbatively,

$$\mathrm{Tr}\ln(\Box+v(x)) = \mathrm{Tr}\ln\left[\Box\left(1+\frac{1}{\Box}v(x)\right)\right] = \mathrm{Tr}\ln\Box+\mathrm{Tr}\left[\frac{1}{\Box}v(x)-\frac{1}{2}\frac{1}{\Box}v(x)\frac{1}{\Box}v(x)+...\right]. \tag{41}$$

The first term above gets canceled by the normalization factor in (39), while the second term can be computed making use of

$$\langle x|\frac{1}{\Box}|y\rangle = i\Delta_F(x-y)\,. \tag{42}$$

Then, at first order,

$$\int d^4x\langle x|\mathrm{Tr}\frac{1}{\Box}v(x)|x\rangle = i\int d^4x\Delta_F(x-x)v(x) = 0\,, \tag{43}$$

where we made used that in dimensional regularization $\Delta_F(0)$ vanishes,

$$\Delta_F(0) = \int\frac{d^4l}{(2\pi)^d}\frac{1}{l^2-i\varepsilon}\sim\frac{1}{4-d}\to 0\,. \tag{44}$$

This contribution corresponds to the tadpole Feynman graphs. Then, at second order, one gets,

$$\frac{1}{2}\int d^4x\langle x|\mathrm{Tr}\left(\frac{1}{\Box}v(x)\frac{1}{\Box}v(x)\right)|x\rangle = \frac{i^2}{2}\int d^4xd^4y\Delta_F(x-y)v(y)\Delta_F(y-x)v(x)\,. \tag{45}$$

This contribution represents the loop correction into the photon propagator, see (34). Next we go to the Lorentz gauge $\partial^\mu A_\mu = 0$ and use the representation

$$\Delta_F(x-y)\partial_\mu\partial_\nu\Delta_F(x-y) = (d\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\frac{\Delta_F^2(x-y)}{4(d-1)}\,. \tag{46}$$

Then, after some integration by parts we obtain the 1-loop effective Lagrangian for the gauge field,

$$\begin{aligned} \Delta\mathcal{L} &= -\frac{1}{2}\int d^4x\langle x|\mathrm{Tr}\left(\frac{1}{\Box}v(x)\frac{1}{\Box}v(x)\right)|x\rangle \\ &= -e^2\int d^4xd^4yF_{\mu\nu}(x)\frac{\Delta_F^2(x-y)}{4(d-1)}F^{\mu\nu}(y)\,. \end{aligned} \tag{47}$$

Then we evaluate $\Delta_F^2(x-y)$ in Dim. Reg:

$$\begin{aligned}\Delta_F^2(x-y) &= \int \frac{d^dk}{(2\pi)^d} e^{ik(x-y)} \left(-\frac{1}{16\pi^2}\right)\left[\frac{2}{4-d} - \gamma + \ln 4\pi - \ln(k^2/\mu^2)\right] \\ &= \left(-\frac{1}{16\pi^2}\right)\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]\delta_D^{(4)}(x-y) + \frac{1}{16\pi^2}L(x-y)\,,\end{aligned} \tag{48}$$

where the first (local) contribution stands for the divergent part and the last contribution denotes the Fourier transform of the finite part $\sim \ln(k^2/\mu^2)$, which is non-local in space. Putting all together, the one loop effective action for the gauge field takes the following form,

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}Z_3'^{-1} + \beta e^2 \int d^4x d^4y F_{\mu\nu}(x)L(x-y)F^{\mu\nu}(y)\,, \tag{49}$$

where β denotes the beta function, and Z'^{-1} is the wavefunction renormalization constant.

The above result can be generalized to an arbitrary set of fields and non-abelian gauge field theories. The only one step which is non-trivial is the introduction of the Faddeev-Popov ghosts, which we discuss now in detail.

2.1.2 Faddeev-Popov ghosts

Formally, integral likes $\int \mathcal{D}A_\mu$ are to be performed over all configurations of A, including the ones that are equivalent up to a gauge transformation. Thus, we integrate over an infinite set of copies of just one configuration. Thus, the choice of the measure $\mathcal{D}A_\mu$ seems to miss the information about the gauge invariance. The Faddeev-Popov method is aimed at fixing the correct integration measure in the partition function of gauge theories.

As an example, we start with the abelian gauge theory with the transformation rule,

$$A_\mu \to A_\mu^{(\theta)} = A_\mu + \partial_\mu\theta\,, \tag{50}$$

and the gauge condition which can be expressed in the form,

$$f(A_\mu) = F(x)\,. \tag{51}$$

The Faddeev-Popov method amounts to inserting the identity,

$$\begin{aligned}1 &= \int \mathcal{D}\theta\, \delta_D(f(A_\mu^{(\theta)}) - F)\Delta(A), \quad \text{where} \\ \Delta(A) &\equiv \det\left(\frac{\partial f}{\partial\theta}\right)\,,\end{aligned} \tag{52}$$

in the partition function. Note that $\Delta(A)$ is called the Faddeev-Popov determinant and it is, in general, independent of θ. The partition function then takes the following form:

$$Z = \mathcal{N}'^{-1} \int \mathcal{D}\theta \mathcal{D}A_\mu \, \delta_D(f(A_\mu^{(\theta)}) - F(x))\Delta(A)e^{iS} \,. \tag{53}$$

Since the above expression does not depend on $F(x)$, we can use another trick and multiply it by a unity obtained from the Gaussian integral over F,

$$1 = N(\xi) \int \mathcal{D}F e^{-\frac{i}{2\xi}\int d^4x F(x)^2} \,, \tag{54}$$

where $N(\xi)$ is a constant. Inserting this into our partition function yields,

$$Z = \mathcal{N}'^{-1} N(\xi) \int \mathcal{D}\theta \mathcal{D}A_\mu \mathcal{D}F \; \delta_D(f(A_\mu^{(\theta)}) - F(x))\Delta(A)e^{iS - \frac{i}{2\xi}\int d^4x F(x)^2} \,. \tag{55}$$

Performing the integral over θ and $F(x)$ we get,

$$Z = \mathcal{N}^{-1} \int \mathcal{D}A_\mu \; \Delta(A)e^{iS - \frac{i}{2\xi}\int d^4x f(A_\mu)^2} \,. \tag{56}$$

The piece $\frac{i}{2\xi}\int d^4x f(A_\mu)^2$ above is the familiar gauge fixing term.

The Faddeev-Popov determinant can be expressed as an integral over an artificial fermion field c,

$$\Delta(A) = \det\left(\frac{\partial f}{\partial\theta}\right) = \int \mathcal{D}c\mathcal{D}\bar{c} \exp\left\{i\int d^4x \; \bar{c}\frac{\partial f}{\partial\theta}c\right\} \,. \tag{57}$$

This field is called the ghost field, it does not correspond to any physical asymptotic states; it appears only inside loops in calculations. In QED $\partial f/\partial\theta$ in independent on A_μ, thus the Faddeev-Popov determinant is just a constant and can be dropped. In the non-abelian case, however, ghosts cannot be neglected and moreover, are essential for a correct quantization.

In conclusion, it should be pointed out that the results obtained in our toy model of scalar QED can be generalized to a more generic setup. For instance, in the case of the theory with the following "bare" Lagrangian with the background field gauge field Γ,

$$\mathcal{L} = \phi^*[d_\mu d^\mu + \sigma(x)]\phi - \frac{\Gamma_{\mu\nu}^2}{4} \,, \tag{58}$$

where $\phi = (\phi_1, ...)$ is some multiplet,

$$\begin{aligned} d_\mu &= \partial_\mu + \Gamma_\mu(x) \,, \\ \Gamma_{\mu\nu} &= \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] \,, \end{aligned} \tag{59}$$

the 1-loop correction to the "bare" action reads,

$$\Delta S = \int d^4x d^4y \; \mathrm{Tr}\left[\Gamma_{\mu\nu}(x)\frac{\Delta_F^2(x-y)}{4(d-1)}\Gamma_{\mu\nu}(y) + \frac{1}{2}\sigma(x)\Delta_F^2(x-y)\sigma(y)\right] \,. \tag{60}$$

Thus, the divergences are local,

$$\Delta S_{div} = \int d^4x \, \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi \right) \mathrm{Tr} \left[\frac{1}{12} \Gamma^2_{\mu\nu}(x) + \frac{1}{2} \sigma(x)^2 \right] . \tag{61}$$

To sum up, the main advantages of the background field method are,

- it deals directly with the action
- it retains symmetries
- it makes the renormalization of nonlinear field theories easy
- it allows to account for many scattering amplitudes at once

The background field and the "quantum" field can coincide and yet the formalism will work in a completely similar manner. One just has to formally decompose this field into the background and the "quantum" modes,

$$\phi = \bar{\phi} + \delta\phi \, . \tag{62}$$

2.2 Background field method in GR

We start to compute the 1-loop effective action in GR by decomposing the metric into the backround and quantum pieces as discussed above,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \, . \tag{63}$$

This decomposition will be considered as exact, i.e. for the inverse metric we have,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h^{\nu}_{\lambda} + \dots \, . \tag{64}$$

In what follows the indices will be raised and lowered using the background metric $\bar{g}_{\mu\nu}$. Now we straightforwardly expand the connection and the Ricci scalar,

$$\begin{aligned} \Gamma^{\mu}_{\nu\rho} &= \bar{\Gamma}^{\mu}_{\nu\rho} + \Gamma^{\mu\,(1)}_{\nu\rho} + \Gamma^{\mu\,(2)}_{\nu\rho} + \dots \, , \\ R &= \bar{R} + R^{(1)} + R^{(2)} + \dots \, , \end{aligned} \tag{65}$$

where we used the notation emphasizing the power counting $R^{(n)} = O(h^n)$. It should be stressed that all terms in this expansion are manifestly covariant w.r.t $\bar{g}_{\mu\nu}$, e.g.,

$$\Gamma^{\mu\,(1)}_{\nu\rho} = \frac{1}{2} \bar{g}^{\mu\lambda} [\bar{D}_{\nu} h_{\lambda\rho} + \bar{D}_{\rho} h_{\nu\lambda} - \bar{D}_{\lambda} h_{\nu\rho}] \, , \tag{66}$$

which displays the gauge invariance of the formalism at each step. Notice that the gauge transformations $x^{\mu} \to x^{\mu} + \xi^{\mu}(x)$ imply the following change of the quantum metric h,

$$h'_{\mu\nu} = h_{\mu\nu} + \bar{D}_{\mu} \xi_{\nu} + \bar{D}_{\nu} \xi_{\mu} \, . \tag{67}$$

The net result of our expansion is,

$$\mathcal{L} = -\frac{2}{\kappa^2}\sqrt{-g}R = \sqrt{-\bar{g}}\Big[-\frac{2}{\kappa^2}\bar{R} - \frac{1}{\kappa}\left(h\bar{R} - 2\bar{R}^{\alpha}_{\nu}h^{\nu}_{\alpha}\right) \\ + \frac{1}{2}\bar{D}_{\alpha}h_{\mu\nu}\bar{D}^{\alpha}h^{\mu\nu} - \frac{1}{2}\bar{D}_{\alpha}hD^{\alpha}h + \bar{D}_{\nu}h\bar{D}^{\beta}h^{\nu}_{\beta} - \bar{D}_{\nu}h_{\alpha\beta}\bar{D}^{\alpha}h^{\nu\beta} \\ - \bar{R}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha}_{\beta}h^{\beta}_{\alpha}\right) + hh^{\alpha}_{\nu}\bar{R}^{\nu}_{\alpha} + 2h^{\nu}_{\beta}h^{\beta}_{\alpha}\bar{R}^{\alpha}_{\nu}\Big] \,, \tag{68}$$

where we denote $h_{\mu\nu}\bar{g}^{\mu\nu} \equiv h$. The term linear in $h_{\mu\nu}$ vanishes by the equations of motion. Now let's fix the gauge. The generalization of the de Donder gauge for a generic background can be obtained by changing partial derivatives to covariant ones,

$$\bar{D}^{\mu}h_{\mu\nu} - \frac{1}{2}\bar{D}_{\nu}h = 0\,. \tag{69}$$

The gauge fixing term in the action reads,

$$\mathcal{L}_{g.f.} \equiv \frac{1}{2}C_{\nu}C^{\nu} = \frac{1}{2}\left[\bar{D}^{\mu}h_{\mu\nu} - \frac{1}{2}\bar{D}_{\nu}h\right]^2 . \tag{70}$$

Notice that the quantity C_{ν} transforms under the gauge transformations as,

$$\begin{aligned} C'_{\nu} =& C_{\nu} + \bar{D}^{\mu}(\bar{D}_{\nu}\xi_{\mu} + \bar{D}_{\mu}\xi_{\nu}) - \bar{D}_{\nu}\bar{D}_{\mu}\xi^{\mu} \\ =& C_{\nu} + \bar{D}^{\mu}\bar{D}_{\mu}\xi_{\nu} - [\bar{D}_{\nu}, \bar{D}_{\mu}]\xi^{\mu}\,, \\ =& C_{\nu} + \left(\bar{g}_{\mu\nu}\bar{D}^2 + \bar{R}_{\mu\nu}\right)\xi^{\nu}\,. \end{aligned} \tag{71}$$

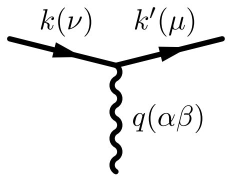
The last missing step is the inclusion of Faddeev-Popov ghosts. In fact, Feynman was first to introduce artificial particles in order that the optical theorem be true in quantum gravity. He called them "dopey particles". The reader is advised to consult Ref. [2] for an amusing conversation between DeWitt and Feynman at the conference when the "dopey particles" were introduced.

Since in gravity the gauge fixing condition has a vector form, the ghosts have to be "fermionic vectors".[2] Introducing ghosts along the lines of (57) and using Eq. (71) we get,

$$\begin{aligned} \det\frac{\partial C_{\nu}}{\partial\xi_{\mu}} =& \det\left[\bar{g}_{\mu\nu}\bar{D}^2 + \bar{R}_{\mu\nu}\right] \\ =& \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\ \exp\left\{i\int d^4x\sqrt{-g}\ \bar{\eta}^{\mu}(\bar{g}_{\mu\nu}\bar{D}^2 + \bar{R}_{\mu\nu})\eta^{\nu}\right\}\,. \end{aligned} \tag{72}$$

The action above implies the following Feynman rules upon flat space,

[2] Recall that in the Yang-Mills theories the ghosts are "fermionic scalars".



$$= -\frac{i\kappa}{2}\left[\eta_{\mu\nu}k_\alpha k'_\beta + \eta_{\mu\nu}k_\beta k'_\alpha - \eta_{\mu\alpha}q_\beta k'_\nu - \eta_{\mu\beta}q_\alpha k'_\nu\right]. \tag{73}$$

Identifying the fields from general expressions (58) and (59) with the background and quantum metrics (68), one can readily obtain the expression for the 1-loop effective action in GR. This was done for the first time by 't Hooft and Veltman [1]. We will show this result in a moment using a different technique: heat kernel.

In summary, we have shown that in the background field method the partition function for quantum gravity reads,

$$Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\eta\mathcal{D}\bar{\eta}\mathcal{D}\phi\ \exp\left\{i\int d^4x\sqrt{-g}[\mathcal{L}(h) + \mathcal{L}_{gf}(h) + \mathcal{L}_{ghosts}(\eta,\bar{\eta},h) + \mathcal{L}_{matter}(h,\phi)]\right\}, \tag{74}$$

where by ϕ we denoted matter fields.

3 The Heat Kernel method

3.1 General considerations

The Heat Kernel is an extremely useful tool widely used in many areas of physics and mathematics. Its application in Quantum Field Theory (QFT) started from the paper by Fock [3] and Schwinger [4] who noticed that the Green functions can be represented as integrals over auxiliary "proper time" variable. Later, DeWitt made the heat kernel technique the powerful tool of computing one–loop divergences in Quantum Gravity in the manifestly covariant approach [5]. Here we will just sketch the main idea, meaning its application to Quantum Gravity. An extensive review of the technique with examples in various areas of physics can be found, e.g, in [6].

Let D be a self–adjoint differential operator in d dimensions [3]. Consider the function

$$G(x,y,\tau,D) = \langle x|e^{-\tau D}|y\rangle. \tag{75}$$

It obeys the following relations,

$$\frac{\partial}{\partial\tau}G(x,y,\tau,D) = -DG(x,y,\tau,D), \tag{76}$$

[3] With respect to a suitable scalar product.

$$G(x,y,0,D) = \delta(x-y). \tag{77}$$

One can combine the last two properties and write

$$(\partial_\tau + D)G(x,y,\tau,D) = \delta(x-y)\delta(\tau). \tag{78}$$

Hence one recognizes in G the Green function of the operator $\partial_\tau + D$,

$$G(x,y,\tau,D) = \langle x,\tau|\frac{1}{\partial_\tau + D}|y,0\rangle. \tag{79}$$

For example, if $D = \alpha\triangle$ with α being some constant, then G is the Green function of the heat equation, hence the name. Consider now $D = D_0$, where

$$D_0 = \Box + m^2, \quad \Box = -\partial_\tau^2 + \triangle. \tag{80}$$

Straightforward calculations lead to

$$G_0 \equiv G(x,y,\tau,D_0) = \frac{1}{(4\pi\tau)^{d/2}} e^{-i\left(\frac{(x-y)^2}{4\tau} + \tau m^2\right)}. \tag{81}$$

As a simple example of the use of G, let us compute the Feynman propagator in the theory of the scalar field in 4 dimensions. Using the equality

$$\frac{i}{A+i\epsilon} = \int_0^\infty d\tau e^{i\tau(A+i\epsilon)}, \tag{82}$$

we have

$$iD_F(x-y) = \langle x|\frac{i}{\Box + m^2 + i\epsilon}|y\rangle = -i\int_0^\infty \frac{d\tau}{16\pi^2\tau^2} \exp\, i\left[\frac{(x-y)^2}{4\tau} + \tau(m^2 + i\epsilon)\right]. \tag{83}$$

In the limit $m = 0$, the last expression turns to

$$iD_F(x-y) = -\frac{1}{4\pi}\frac{1}{(x-y)^2 - i\epsilon}, \tag{84}$$

and coincides with the standard result.

As was said before, the particular usefulness of the heat kernel method in QFT is related to computation of one–loop divergences. Recall that quantum effects due to background fields are contained in the one–loop effective action

$$W \sim \ln\det D. \tag{85}$$

Using the integral

$$\ln\frac{a}{b} = \int_0^\infty \frac{d\tau}{\tau}\left(e^{-\tau a} - e^{-\tau b}\right), \tag{86}$$

from (85) and (75) we have

$$W \sim \int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}G(x,x,\tau,D) + C = \mathrm{Tr}'\int_0^\infty \frac{d\tau}{\tau}\int d^dx\langle x|e^{-\tau D}|x\rangle + C. \tag{87}$$

Here C is some constant, and by Tr' we understand the trace taken over internal indexes of D.

In general, the expression (87) can be divergent at both limits of integration. Those corresponding to large τ are IR–divergences, and we will not consider them here. Rather, we will be interested in UV–divergences which appear in the limit $\tau \to 0$. Therefore, we need to know the asymptotic behavior of G at small τ. The latter is given by

$$G(x, y, \tau, D) = G(x, y, \tau, D_0)(a_0 + a_1\tau + a_2\tau^2 + ...), \tag{88}$$

where $a_i = a_i(x, y)$ are local polynomials of the background fields. Substituting (88) into (87) gives

$$\mathrm{Tr}\ln D = -\frac{i}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} m^{d-2n} \Gamma(n - \frac{d}{2}) \mathrm{Tr}' a_n(x). \tag{89}$$

3.2 Applications

Now we are going to compute G explicitly for quite generic form of D,

$$D = d_\mu d^\mu + \sigma(x), \quad d_\mu = \partial_\mu + \Gamma_\mu(x). \tag{90}$$

Inserting the full set of momentum states, one can rewrite G as

$$G(x, x, \tau, D) = \langle x | e^{-\tau D} | x \rangle = \int \frac{d^d p}{(2\pi)^d} e^{-ipx} e^{-\tau D} e^{ipx}, \tag{91}$$

where we have used the following normalizations,

$$\langle p | x \rangle = \frac{1}{(2\pi)^{d/2}} e^{ipx}, \quad \langle x | x' \rangle = \delta^{(d)}(x - x'), \quad \langle p | p' \rangle = \delta^{(d)}(p - p'). \tag{92}$$

Using the relations

$$d_\mu e^{ipx} = e^{ipx}(ip_\mu + d_\mu), \quad d_\mu d^\mu e^{ipx} = e^{ipx}(ip_\mu + d_\mu)(ip^\mu + d^\mu), \tag{93}$$

we derive

$$\begin{aligned} G(x, x, \tau, D) &= \int \frac{d^d p}{(2\pi)^d} e^{-\tau\left[(ip_\mu + d_\mu)^2 + m^2 + \sigma\right]} = \\ &= \int \frac{d^d p}{(2\pi)^d} e^{\tau(p^2 - m^2)} e^{-\tau(d\cdot d + \sigma + 2ip\cdot d)}. \end{aligned} \tag{94}$$

We observe that the first exponential in (94) corresponds to the free theory result, while all the interesting physics is contained in the second exponential. The latter can be expanded in powers of τ. Integrating over p gives (for the details of calculations, see Appendix B of [7]),

$$G(x, x, \tau, D) = \frac{i e^{-m^2\tau}}{(4\pi\tau)^{d/2}} \left[1 - \sigma\tau + \tau^2 \left(\frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]] \right) \right]. \tag{95}$$

Comparing with (88), we have

$$a_0 = 1, \quad a_1 = -\sigma, \quad a_2 = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]]. \quad (96)$$

As an application of the result derived above, consider the scalar QED. We have

$$d_\mu = \partial_\mu + ieA_\mu, \quad m = 0, \quad \sigma = 0, \quad [d_\mu, d_\nu] = ieF_{\mu\nu}. \quad (97)$$

Hence the coefficients (96) are

$$a_1 = 0, \quad a_2 = \frac{1}{12}F_{\mu\nu}F^{\mu\nu}. \quad (98)$$

It then follows that the divergent part of the 1–loop effective action is

$$S_{div} = \int d^4x \frac{1}{\epsilon}\frac{e^2}{16\pi}\frac{1}{12}F_{\mu\nu}F^{\mu\nu}. \quad (99)$$

As a second example, consider the renormalization of the scalar field in the presence of background gravitational field. We specify the theory as follows,

$$\mathcal{L} = \xi R\varphi^2 + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2, \quad (100)$$

where ξ is a non–minimal coupling constant. In this case straightforward calculations give

$$a_1 = \left(\frac{1}{6} - \xi\right)R, \quad a_2 = \frac{1}{180}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}(6\xi - 1)^2 R^2 - 6\Box R\right). \quad (101)$$

The divergent part of the effective action is then given by

$$S_{div} = \int d^4x\sqrt{-g}\frac{1}{\epsilon}\frac{1}{180}\frac{1}{16\pi^2}\left[3\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{8}R^2\right) + \frac{5}{2}(6\xi - 1)R^2\right]. \quad (102)$$

3.3 Gauss–Bonnet term

Topological properties of manifolds are captured by invariant combinations of local quantities. In case of even–dimensional boundaryless manifold one of such invariants is the Euler characteristic χ given by

$$\chi = \int d^4x\sqrt{-g}E, \quad E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (103)$$

Adding this term to the action does not affect the equations of motion, since E can be written as a divergence of a topological current,

$$\sqrt{-g}E = \partial_\mu J^\mu, \quad J^\mu = \sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma}{}^{\kappa\lambda}\Gamma^\rho_{\kappa\nu}\left(\frac{1}{2}R^\sigma_{\lambda\rho\sigma} + \frac{1}{3}\Gamma^\sigma_{\tau\rho}\Gamma^\tau_{\lambda\sigma}\right). \quad (104)$$

Consequently, whenever one has a bilinear combination of Riemann, Ricci or scalar curvature tensors, one can eliminate one of them by the means of the Gauss–Bonnet term [4].

3.4 Pure gravity limit

Now we are going to see how the gravity itself renormalizes in the presence of external gravitational field. The functions $a_{i,grav.}$ in the short–time expansion (88) are called DeWitt-Seeley-Gilkey coefficients. Computation of the second coefficient gives the following result,

$$a_{2,grav.} = \frac{215}{180}R^2 - \frac{361}{90}R_{\mu\nu}R^{\mu\nu} + \frac{53}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \\ = \frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}, \qquad (105)$$

where in the second line we have made use of the Gauss–Bonnet term (103). From (105) an interesting feature of pure gravity in four dimensions follows. Recall that in the absence of matter Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \qquad (106)$$

Hence the solution is $R_{\mu\nu} = 0$ and $R = 0$. But then $a_{2,grav} = 0$, and we arrive at conclusion that pure gravity can be made finite at one–loop by a suitable fields redefinition. This nice property is, however, peculiar to the specific theory. First, only in four dimensions one can use the Gauss–Bonnet term (103) to make the divergent term vanish at one loop. For example, in six dimensions pure gravity diverges at one loop. Second, the real world contains the matter which spoils the one–loop finiteness. Third, even for pure gravity the renormalizability does not hold anymore when one goes to higher loops. For example, the two–loop calclulation reveals the following behavior of the divergent part of the action [8]

$$S_{2,div} = \int d^4x\sqrt{-g}\frac{1}{\epsilon}\frac{209}{2880}\frac{\kappa^2}{(16\pi^2)^2}R^{\mu\nu\alpha\beta}R_{\alpha\beta\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}, \qquad (107)$$

and this divergence cannot be canceled by the renormalization of the Einstein–Hilbert action.

To summarize, we have seen that the heat kernel method is a powerful and universal tool of computing one–loop divergences of the effective action. In particular,

- it is easy to apply,

[4] Note that if E is coupled to other fields, e.g., through the terms $f(\varphi)E$, it does contribute to the equations of motion.

- it captures the divergent parts of all one–loop diagrams,
- it offers the manifestly covariant approach.

On the other hand, the heat kernel method

- does not capture the finite ln q^2 parts,
- is not applicable beyond the one–loop approximation.

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