Last section - end of Lecture 4

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1 An Introduction to Non-local Effective Actions

In this final segment, I would like to describe some aspects of the gravitational effective field theory which need to be developed more fully in the future. We have seen how to quantize the theory and make quantum field theoretic predictions within general relativity. The most straightforward amplitudes to calculate are scattering matrix elements - this is what quantum field theory does well. But most applications of general relativity are not scattering amplitudes. In order to address quantum effects more generally one needs to be able to treat the non-linear classical solutions. One way to address such settings is to use non-local effective actions expressed using the curvatures.

Why use an effective action? While most quantum calculation are done in momentum space, for general relativity it is best to work in coordinate space. In particular, we know how to write the curvatures and covariant derivatives in terms of the field variables. Using an effective action allows one to summarize quantum effects in a generally covariant fashion.

Why non-local? As has been stressed here, locality is the key to the effective field theory treatment, as non-local effects correspond to long distance propagation and hence to the reliable predictions at low energy. The local terms by contrast summarize - in a few constants - the unknown effects from high energy. Having both local and non-local terms allow us to implement the effective field theory program using an action built from the curvatures.

1.1 Anomalies in General

My starting point may seem a bit unexpected, but I would like to begin by a discussion of anomalies. We are used to thinking of an anomalies as a UV phenomenon. For example, in a path integral context, anomalies can be associated with the non-invariance of the path integral measure [1]. This is regularized by adding a UV cutoff, and finding finite effects as the cutoff is removed.

Superficially this should bother an effective field theorist. If the anomaly can only be found by treating the UV sector of the theory, how can we be sure about it as we do not have complete knowledge about UV physics? Could we change something about the high energy part of the theory and make the anomaly go away? What has happened to the argument that UV effects are local and are encoded in local effective Lagrangians?

But there is also an IR side to anomalies. For example, both the axial anomaly and the trace anomaly can be uncovered by dispersion relations, with the dominant contributions coming from low energy[2, 3, 4]. And direct calculation can reveal non-local effect actions which encode the predictions of the anomalies.

Indeed, we have already seen one such example. In the section on the background field method, I calculated the effect of integrating out a massless scalar field coupled to photons. After renormalization, the result was an effective action of the form

$$S = -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} + \beta e^2 \int d^4x d^4y\ F_{\mu\nu}(x) L(x-y) F^{\mu\nu}(y) \quad , \tag{1}$$

where the function $L(x-y)$ is the Fourier transform of $\log q^2$,

$$L(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \ln\left(\frac{-q^2}{\mu^2}\right) \tag{2}$$

Using the notation

$$L(x-y) \equiv \langle x| \ln\left(\frac{\Box}{\mu^2}\right) |y\rangle \tag{3}$$

and making a conventional rescaling of the photon field, this non-local action can be put in the form

$$S = \int d^4x\ \ -\frac{1}{4}F_{\rho\sigma}\left[\frac{1}{e^2(\mu)} - b\ln\left(\Box/\mu^2\right)\right]F^{\rho\sigma} \quad . \tag{4}$$

One sees immediately the connection of this action to the running of the electric charge, with b being related to the beta function.

The fundamental action for QED with massless particles is scale invariant, i.e. it is invariant under the transformations $A_\mu(x) \to \lambda A_\mu(\lambda x)$, $\psi(x) \to \lambda^{3/2}\psi(\lambda x)$, $\phi(x) \to \lambda\phi(\lambda x)$. We can define an associated concerved current $J_\mu = T_{\mu\nu}x^\nu$ with the conservation condition $\partial^\mu J_\mu = 0$ implying the tracelessness of $T_{\mu\nu}$, $T^{\mu}_{\mu} = 0$. However, the scale symmetry has an anomaly, and after quantum corrections the trace does not vanish.

An infrared demonstration of this can come from the non-local effective action derived above. Under rescaling we have

$$L(x-y) = \lambda^4 L(\lambda x - \lambda y) + \ln \lambda^2 \delta^4(x-y) \tag{5}$$

and the rescaling is no longer a symmetry of the quantum action. Using this, one readily finds (in the conventional normalization) the trace anomaly relation

$$T^{\nu}_{\ \nu} = \frac{be^2}{2} F_{\rho\sigma} F^{\rho\sigma} \quad . \tag{6}$$

The relation of the anomaly to the running coupling is apparent. The trace anomaly cannot be derived from any gauge invariant local action, but it does follow from the calculated non-local effective action.

1.2 Conformal Anomalies in Gravity

The couplings of massless particles to gravity can have a conformal symmetry which is similar to the scale symmetry described above. This involves the local transformation

$$g'_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x) \qquad\qquad \phi'(x) = e^{-p\sigma(x)} \phi(x) \tag{7}$$

with $p = 1$ for scalar fields, $p = 0$ for gauge fields and $p = 3/2$ for fermions. With massless scalars there needs to be an extra term in the action $-R\phi^2/6$ in order to have conformal symmetry, but for massless fermions and gauge field the symmetry is automatic. When this is a symmetry of the matter action S_m, one must have

$$\delta S_m = 0 = \left[\frac{\delta S_m}{\delta \phi} \delta\phi + \frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right] \quad . \tag{8}$$

The first term here vanishes by the matter equation of motion. In the second one, the variation with respect to $g_{\mu\nu}$ gives the energy momentum tensor, and $\delta g_{\mu\nu} = 2\sigma(x) g_{\mu\nu}$, so that the condition of conformal invariance requires $T^{\mu}_{\mu} = 0$. The gravitational part of the action is itself not conformally invariant, as $R' = e^{2\sigma}[R + 6\Box\sigma]$.

However, the conformal symmetry of the massless matter sector is anomalous. In the path integral treatment this can be traced to the Jacobean of the transformation. This can be regularized in an invariant way using the heat kernal expansion. For the scalar field transformation of Eq. 7 we have

$$\mathcal{J} = \det[e^{-\sigma}] = \lim_{M\to\infty} \exp\left[\mathrm{Tr}\log\left(-\sigma e^{-D^2/M^2}\right)\right] = \exp\left[-\sigma a_2(x)\right] \quad . \qquad (9)$$

The consequence of this non-invariance can be translated into an anomalous trace

$$T^\mu_\mu = \frac{1}{16\pi^2}a_2 = \frac{1}{16\pi^2}\frac{1}{18}\left[R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \Box R\right] \qquad (10)$$

The expression in terms of a_2 is generic, and the second form is specific to scalar fields. Much more detail about the conformal anomaly can be found in the books by Birrell and Davies [5] and by Parker and Toms [6].

1.3 Non-local Effective Actions

Deser Duff and Isham [7] were the first to argue that the conformal anomaly was connected to a non-local effective action. Having seen the QED example in the previous section, this should not surprise us. However, the importance of the effective action technique goes well beyond just anomalies. It allows the low energy quantum effects to be summarized in a covariant fashion. This latter aspect has been developed especially by Barvinsky, Vilkovisky and collaborators (here called BV) [8, 9, 10]. The presentation here is only introductory.

The basic idea of the BV program is to express one loop amplitudes in terms of curvatures and covariant derivatives. For example, much like the QED example above we could expect a term of the form

$$\int d^4x\sqrt{-g}\ R\log\nabla^2 R \qquad (11)$$

where $\log\nabla^2$ is a covariant object which reduces to $\log\Box$ in flat space[1]. Another possible term could be

$$\int d^4x\sqrt{-g}\ R^2\frac{1}{\nabla^2}R \qquad (12)$$

where $1/\nabla^2$ represents the covariant massless scalar propagator. We note that both of the terms just mentioned are of the same order in the derivative expansion.

One loop Feynman diagrams can be expressed in terms of scalar bubble, triangle and box diagrams. The bubble diagram is UV divergent, and we

[1] The discussion of possible forms for $\log\nabla^2$ is too extensive for the present context.

have seen how the heat kernel method encodes these divergences in terms of the curvatures. Along with the divergence comes a factor of $\log q^2$ in momentum space which becomes $\log \nabla^2$ in the non-local effective action. From this we see that the terms of order $R \log \nabla^2 R$ come with coefficients which are fixed from the one-loop divergences (as was true in the QED example also). These can be calculated in a non-local version of the heat kernel method [8, 11], or simply matched to the perturbative one-loop calculations [12]. The results, taken from Ref. [12] in two different bases are

$$S_{NL} = \int d^4x \sqrt{g} \left(\alpha R \log\left(\frac{\Box}{\mu_\alpha^2}\right) R + \beta R_{\mu\nu} \log\left(\frac{\Box}{\mu_\beta^2}\right) R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} \log\left(\frac{\Box}{\mu_\gamma^2}\right) R^{\mu\nu\alpha\beta} \right) \tag{13}$$

or

$$S_{NL} = \int d^4x \, \sqrt{g} \, \Big[\bar{\alpha} R \log\left(\frac{\Box}{\mu_1^2}\right) R + \bar{\beta} C_{\mu\nu\alpha\beta} \log\left(\frac{\Box}{\mu_2^2}\right) C^{\mu\nu\alpha\beta} \\ + \bar{\gamma} \big(R_{\mu\nu\alpha\beta} \log(\Box) R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} \log(\Box) R^{\mu\nu} + R \log(\Box) R \big) \Big] . \tag{14}$$

Here the coefficients of the various terms are displayed in Table 1. In the second version, $C_{\mu\nu\alpha\beta}$ is the Weyl tensor

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} - \frac{1}{2}\left(R_{\mu\alpha} g_{\nu\beta} - R_{\nu\alpha} g_{\mu\beta} - R_{\mu\beta} g_{\mu\alpha} + R_{\nu\beta} g_{\mu\alpha}\right) \\ &+ \frac{R}{6}\left(g_{\mu\alpha} g_{\nu\beta} - e_{\nu\alpha} e_{\mu\beta}\right) \quad . \end{aligned} \tag{15}$$

The second form also emphasizes a useful point. As described previously, the local Lagrangian comes with two independent terms, because the Gauss-Bonnet identity tells us that one combination of curvatures is a total derivative. The non-local action can have three terms because that third curvature combination can have non-trivial effects when the non-local function $\log \nabla^2$ occurs between the curvatures. The two coefficients in the local action include functions of the renormalization scale in the form $c_i(\mu^2)$. The logarithms also come with a scale factor $\log \mu^2$ which is itself local - $< x|\log \mu^2|y >= \log \mu^2 \ \delta^4(x-y)/\sqrt{-g}$. The total combination is independent of μ. In the second version of the non-local action, the last combination has no μ dependence because the local combination vanishes.

The phenomenology of these non-local actions are just begining to be explored. I did not have time in the lectures to describe these early works, but I can here refer the reader to some examples in [13, 14, 12, 15, 16, 17].

	α	β	γ	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
Scalar	$5(6\xi-1)^2$	-2	2	$5(6\xi-1)^2$	3	-1
Fermion	-5	8	7	0	18	-11
Vector	-50	176	-26	0	36	-62
Graviton	430	-1444	424	90	126	298

Table 1: Coefficients in the non-local action due to different fields. All numbers should be divided by $11520\pi^2$.

The gravitational conformal anomalies have also been uncovered in the non-local actions [18, 10].

At third order in the curvature, very many more terms are possible, having forms similar to Eq. 12. Interested readers are invited to peruse the 194 page manuscript describing these, Ref. [9]. These are so complicated that they will probably never be applied in full generality. However, we eventually will need to understand what type of effect they could have and if there is any interesting physics associated with them.

It is important to be clear that the usual local derivative expansion, which for gravity is also a local expansion in the curvature, is quite different from this non-local expansion in the curvature. In the local expansion, each subsequent term is further suppressed in the energy expansion at low energy. With the non-local expansion, the terms are all technically at the same order in the energy expansion. However, they represent different effects - at the very least representing bubble diagrams vs triangle diagrams. It is expected that there will be settings where the curvature is small that the terms third order in the curvature can be neglected.

1.4 An Explicit Example

Because the gravity case quickly becomes complicated, it is useful to go back to a simpler example in order to get a feel for non-local actions. To do this let us consider the QED example with a massless scalar considered previously but now coupled up to gravity also. This is straightforward to calculate in perturbation theory. With the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and placing the photons on-shell, we find that the linear term in the gravitational field has the form

$$S = \int d^4x\ h^{\mu\nu} \left[b_s \log\left(\frac{\Box}{\mu^2}\right) T^{cl} + \frac{1}{96\pi^2}\frac{1}{\Box}\tilde{T}^s_{\mu\nu} \right] \quad (16)$$

where b_s is the scalar beta function coefficient and the extra tensor structure is given by

$$\tilde{T}^s_{\mu\nu} = \partial_\mu F_{\alpha\beta} \partial_\nu F^{\alpha\beta} + \partial_\nu F_{\alpha\beta} \partial_\mu F^{\alpha\beta} - \eta_{\mu\nu} \partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} \tag{17}$$

Here we see a logarithmic non-locality similar to those that we have already become familiar with. There is also a $1/\Box$ non-locality, which arose from a factor of $1/q^2$ in the momentum space calculation.

Let me not discuss the logarithm here - it is somewhat complicated to put this in covariant form [15, 19]. However the new $1/\Box$ term is simple to understand. If we want to write this in covariant fashion, we note that we are expecting terms which are generically of the form $F^2(1/\Box)R$, with various tensor index contractions. If we write out all possible contributions and expand these to first order in $h_{\mu\nu}$, it turns out that there is a unique matching to the perturbative result. This is We find the following form to be the most informative

$$\Gamma_{NL}[g, A] = \int d^4x \sqrt{g} \left[n_R F_{\rho\sigma} F^{\rho\sigma} \frac{1}{\nabla^2} R + n_C F^{\rho\sigma} F^{\gamma}_{\lambda} \frac{1}{\nabla^2} C_{\rho\sigma\gamma}{}^{\lambda} \right] \quad . \tag{18}$$

where again $C_{\rho\sigma\gamma}{}^{\lambda}$ is the Weyl tensor. The coefficients for a scalar loop involve

$$n_R = -\frac{\beta}{12e}, \quad n_C = -\frac{e^2}{96\pi^2} \quad . \tag{19}$$

where here β is the QED beta function.

We see in this calculation the prototype of what is happening in gravity. If we think of the field strength tensor $F_{\mu\nu}$ as a "curvature", we have curvature-squared terms with a non-local factor of $\log \Box$ and curvature-cubed term with a non-local factor of $1/\Box$. Both come from one loop diagrams. The pure $\log \Box$ comes from bubble diagrams which are also associated with UV divergences. The $1/\Box$ terms come from the scalar triangle diagram. The coefficients of each of all of these are fixed by direct calculation and are not free parameters. To tie up with our starting point for this section, one can show that the scale anomaly is associated with the log terms and the QED conformal anomaly is associated with the $1/\Box$ terms [15]. That the trace relation is identical in both cases comes from the fact that the beta function determines both terms, and indicates a beautiful consistency within the theory.

1.5 Summary

I have chosen to end on this topic because I feel that it is one of the frontiers of the application of quantum field theory to general relativity. If we are to treat quantum corrections in more complicated settings than scattering amplitudes, we need to treat the full non-linear structure of general relativity. The effective actions summarize the quantum effects with full curvatures. However, the applications of these non-local effective actions have been only lightly explored.

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