Lecture 1

22 September 2016

1 Constructing GR as a gauge theory – QFT point of view [1]

1.1 Preliminaries

Suppose that Einstein had never existed. Then, if we wanted to build gravity in quantum field theory (QFT) framework, we would proceed as with theories of other interactions. At classical level, the Newtonian potential acting between two bodies of masses m_1 and m_2 is given by

$$V = -G\frac{m_1 m_2}{r}, \tag{1}$$

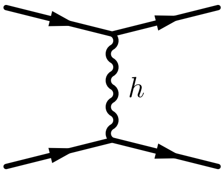
where G is Newton's gravitational constant, $G = M_P^{-2} = (1.22\cdot 10^{19} GeV)^{-1}$, and we use natural units. The law (1) is analogous to that of Coulomb interaction, and we know that the photon field serves as a mediator of electromagnetic interaction. Hence, we can ask: what is the mediator of the gravitational interaction? A little contemplation reveals immediately that this should be a particle of spin 0 or 2. Spin–1 particles are not appropriate since, as we know from electrodynamics, they lead to repulsive as well as attractive force between objects, and we know no examples of repulsive gravity [2]. Higher spin particles cannot be consistently included into QFT framework. The simplest option is, therefore, the Higgs–like force mediated by spin–0 particle. Indeed, consider the interaction of the form

$$\mathcal{L}_{int} \sim \sum_i m_i \left(1 + \frac{h}{v}\right) \bar{\psi}_i \psi_i. \tag{2}$$

Then, one can write the amplitude of interaction as

[1] For early works about gravity as a gauge theory see, e.g., [1],[2].

[2] Except, maybe, that acting at very large distances.



$$= -i\mathcal{M} = \frac{-im_1}{v}\frac{i}{q^2 - m^2}\frac{-im_2}{v}\,, \tag{3}$$

where q is the momentum carrying by the h–particle and m is its mass. From this amplitude, one derives the following potential,

$$V(r) = -\frac{1}{\pi v^2} m_1 m_2 \frac{e^{-mr}}{r}. \tag{4}$$

Taking the limit $m = 0$, we recover the Newtonian potential (1).

There are reasons, however, why this choice of gravity mediator cannot be accepted. First, we know that the bare mass of an object is not a unique source of the gravitational field. For example, the constituent mass of the proton is given by

$$m_p = \langle P|T^\mu_\mu|P\rangle = \langle P|\beta F^2 + m_u \bar{u}u + m_d \bar{d}d|P\rangle, \tag{5}$$

with overall contribution from the quarks being only around $40MeV$, the rest coming from strong interactions represented by the first term in the r.h.s. of (5). Next, in nuclei, binding energy gives an essential contribution to the total mass. Not to mention the massless photon fields that, under this assumption, would not be coupled to gravity at all. Hence we conclude that the source of the gravitational field must be the *total* energy represented by the Energy–Momentum Tensor (EMT) T^{ab} [3].

The second observation is based on Einstein's claim of equality of inertial and gravitational masses, from which the universality of free–fall follows. The latter can be formulated as follows : the pathway of a particle in the gravitational field depends only on the initial position and velocity of that particle. In other words, the geodesic equation does not contain any quantities depending on internal structure of the particle. Furthermore, we recall the second part of Einstein's Equivalence Principle (EP) that claims the physical equivalence of freely falling frames, with its generalization claiming the equivalence of all coordinate frames. The EP implies that for every observer at any moment of proper time one can choose a coordinate frame in which the gravitational field vanishes. Mathematically, this implies the vanishing of the Levi–Civita connection terms $\Gamma^\mu_{\nu\rho}$ [4]. In particular, from the EP it follows that the light must be bent by gravity in the same way as it is bent in accelerating frames. But let us account for this effect within scalar

[3]By small Latin letters we denote the Lorentzian indices.

[4]Note that $\Gamma^\mu_{\nu\rho}$ can be made zero not only at a given point, but also along a given world–line.

gravity framework. Assuming universal coupling — the necessary ingredient for the EP to hold, — the only way to couple the scalar field ϕ to the EMT is through the term of the form

$$\mathcal{L}_{int} \sim \phi T^a_a. \tag{6}$$

But for the electromagnetic field $T^a_a \sim E^2 - B^2 = 0$. Hence, scalar gravity cannot obey the Einstein's EP. We arrive at conclusion that gravity must be mediated by the spin–2 field, and T^{ab} must be the source of that field.

Before exploring this possibility, let us remind some basic properties of EMT. As an example, consider the theory of the real massive scalar field, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - \frac{1}{2}m^2\phi^2. \tag{7}$$

The translational invariance of (7) implies the existence of a conserved current,

$$T_{ab} = \frac{\partial\mathcal{L}}{\partial\partial_a\phi}\partial_b\phi - \eta_{ab}\mathcal{L}, \tag{8}$$

or, explicitly,

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}\left(\eta_{ab}\eta^{cd}\partial_c\phi\partial_d\phi - m^2\phi^2\right). \tag{9}$$

It then follows that on equations of motion $\partial_a T^{ab} = 0$. One can also introduce the charges

$$H = \int d^3x T_{00}, \qquad P_i = \int d^3x T_{0i}, \tag{10}$$

that are time–independent, $\partial_t H = \partial_t P_i = 0$.

Going back to QFT, we derive the following potential for the two body graviton exchange,

$$V \sim \frac{1}{2}\frac{\kappa}{2}T_{ab}\frac{1}{4\pi r}\frac{\kappa}{2}T^{ab} \sim \frac{\kappa^2}{32\pi}\frac{m_1 m_2}{r}, \tag{11}$$

where κ is a constant determining the strength of the gravity coupling. In obtaining this result, we have used the following normalization for T^{ab}:

$$\langle p|p'\rangle = 2E\delta^{(3)}(\vec{p}-\vec{p}'), \tag{12}$$

$$\langle p|T_{ab}|p'\rangle = \frac{1}{\sqrt{2E\ 2E'}}\left[(p_a p'_b + p'_a p_b) - \eta_{ab}(p\cdot p' - m^2)\right]. \tag{13}$$

We see that considering EMT as a source and spin–2 field as a mediator of the gravitational interaction is a reasonable suggestion. Now we want to obtain this prescription from the first principles of QFT.

1.2 Gauge theories — short reminder

In the next two subsections we remind some basic properties of Yang–Mills (YM) gauge theories. Our interest in these theories is based on the observation that the gauge field mediates forces between matter fields, and it couples to the currents of the corresponding global symmetry. Since we know from the preceding discussion that EMT is the natural source of gravity, it is tempting to construct gravity as a gauge field resulting from gauging the global symmetry the EMT corresponds to.

1.2.1 Abelian Case

Consider a theory invariant under some (global) symmetry group. As an example, we will use the theory of massive Dirac field ψ with the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{14}$$

It possesses the invariance under the global transformations $\psi \to e^{-i\theta}\psi$, where θ is a constant. Applying Noether's theorem gives the current

$$j^a = \bar{\psi}\gamma^a\psi, \qquad \partial_a j^a = 0, \tag{15}$$

and the charge

$$Q = \int d^3 j_0. \tag{16}$$

Now we want to make (14) invariant with respect to local transformations:

$$\psi \to e^{-i\theta(x)}\psi. \tag{17}$$

The way to do this is to introduce new field A_μ, which is called a gauge field, and rewrite the Lagrangian in the form

$$\mathcal{L} = \bar{\psi}(iD\!\!\!\!/ - m)\psi, \qquad D_a = \partial_a + ieA_a. \tag{18}$$

To ensure the invariance of (18) under (17), the covariant derivative of the field, $D_a\psi$, must transform as

$$D_a\psi \to e^{-i\theta(x)}D_a\psi. \tag{19}$$

In turn, this implies that the gauge fields transforms as

$$A_a \to A_a + \frac{1}{e}\partial_a\theta(x). \tag{20}$$

The next step is to make the gauge field dynamical. To this end, one should introduce a kinetic term for A_a. The latter can be built as a bilinear combination of the field strength tensor,

$$-\frac{1}{4}F_{ab}F^{ab}, \tag{21}$$

where F_{ab} is defined through the relation

$$[D_a, D_b] = ie(\partial_a A_b - \partial_b A_a) = ieF_{ab}. \tag{22}$$

The expression (21) is positive–definite and invariant under the local transformations (17). The modified Lagrangian is written as

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} + \bar{\psi}(i\not{D} - m)\psi, \tag{23}$$

from which we observe that the coupling of the gauge field A_a to fermions takes the form $j^a A_a$. Hence the current (15) acts as a source of the field A_a.

1.2.2 Nonabelian case

As an example of a theory whose symmetry group is non–abelian, consider the field ψ transforming in a fundamental representation of some compact group, say, $SU(N)$ [5]

$$\psi \to U\psi, \qquad U = e^{-i(\omega_0 + \frac{1}{2}\omega_\alpha \lambda^\alpha)}, \tag{24}$$

where λ^α are generators of $SU(N)$ obeying

$$\left[\frac{\lambda^\alpha}{2}, \frac{\lambda^\beta}{2}\right] = if^{\alpha\beta\gamma}\frac{f^\gamma}{2}, \qquad \mathrm{Tr}\left[\frac{\lambda^\alpha}{2}\frac{\lambda^\beta}{2}\right] = \frac{1}{2}\delta^{\alpha\beta}. \tag{25}$$

The Lagrangian (14) is invariant under the transformations (24) as long as all ω^0, ω_α are constant. To promote its invariance to the local transformations,

$$\psi \to U(x)\psi, \tag{26}$$

we introduce the gauge fields A_a^α and covariant derivative D_a,

$$D_a = \partial_a + ig\frac{\lambda^\alpha}{2}A_a^\alpha \equiv \partial_a + ig\mathcal{A}_a, \tag{27}$$

where we use the matrix notation $\mathcal{A}_a = \frac{\lambda^\alpha}{2}A_a^\alpha$. The transformation properties read as follows,

$$D_a\psi \to U(x)D_a\psi, \tag{28}$$

$$\mathcal{A}_a \to U\mathcal{A}_a U^{-1} + \frac{i}{g}(\partial_a U)U^{-1}, \qquad D_a \to UD_aU^{-1}. \tag{29}$$

Dynamics for the fields A_a^α is given by the field strength tensor F_{ab}^α, or, in matrix notation, $\mathcal{F}_{ab}$. It is defined as

$$[D_a, D_b] = ig\mathcal{F}_{ab} = ig\frac{\lambda^\alpha}{2}F_{ab}^\alpha, \tag{30}$$

and the explicit expressions are given by

$$\mathcal{F}_{ab} = \partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a + g[\mathcal{A}_a, \mathcal{A}_b], \tag{31}$$

$$F_{ab}^\alpha = \partial_a A_b^\alpha - \partial_b A_a^\alpha - gf^{\alpha\beta\gamma}A_a^\beta A_b^\gamma. \tag{32}$$

[5] By ψ now we understand N–component row $(\psi_1, ..., \psi_N)^T$.

1.3 Gravitational field from gauging translations

1.3.1 General coordinate transformations

Our goal is to implement the kind of reasoning outlined above in case of gravity. To generate the field mediating the force whose sources are given by EMT, one should gauge the global symmetry the EMT corresponds to, i.e., one should gauge global translations

$$x^a \to x^a + a^a. \tag{33}$$

Hence we consider the local version of (33),

$$x^\mu \to x^\mu + a^\mu(x), \tag{34}$$

which is equivalent to

$$x^\mu \to x'^\mu(x). \tag{35}$$

In other words, the local shifts constitute the most general transformations of coordinate frame, and we will refer to them as General Coordinate Transformations (GCT). We observe the first qualitative difference between gravity and usual YM–theories. In the case of gravity we gauge one of the space–time symmetries of the original theory. This theory is composed of objects with well defined properties under global Poincare transformations. In order to be able to speak about GCT–invariance, one should define how the components of the original theory are transformed under (34). The promotion of the global Poincare group to GCT is trivial for some objects, and non–trivial for others [6]. In the case of space–time coordinates, we just replace the Lorentzian indices $a, b, ...$ with the world indices $\mu, \nu, ...$, meaning that the general transformations of coordinate frames are now admissible on space–time manifold.

Modulo this observation, the procedure of building GCT–invariant theory seems to be fairly straightforward. Let us sketch the important steps here. By the analogy with YM–theories, we define a new field $g_{\mu\nu}$ such that

$$\frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} \sim T_{\mu\nu}. \tag{36}$$

Using this field, we promote the partial derivatives to covariant ones,

$$D_\mu = \partial_\mu + \Gamma_\mu(g), \tag{37}$$

where $\Gamma_\mu(g)$ are some functions of $g_{\mu\nu}$ to be defined later. To make $g_{\mu\nu}$ dynamical field, we introduce the field strength tensor, schematically,

$$[D, D] \sim \mathcal{R}. \tag{38}$$

[6] For example, the spinors can only be defined on the Minkowski background.

Finally, the invariant action is built from the matter action S_m and the action S_g for the field $g_{\mu\nu}$.

Before realizing this program, let us make a brief comment about EMT. As we will find shortly, $g_{\mu\nu}$ is the symmetric tensor field. Canonical EMT, however, need not be symmetric. Hence, to treat EMT as a source of the gravitational field, one should bring it to the symmetric form without spoiling the corresponding conservation law. This can be achieved by the redefinition [3]

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}, \qquad B^{\rho\mu\nu} = -B^{\mu\rho\nu}. \tag{39}$$

It is readily seen that once $\partial_\mu T^{\mu\nu} = 0$, then $\partial_\mu \tilde{T}^{\mu\nu} = 0$ as well. Note that this *is* the modification of the current, although it preserves the on–shell conservation law. Note also that the choice of $B^{\rho\mu\nu}$ tensor is not unique.

Let us start implementing the program outlined above. In special–relativistic field theories whose gauged versions we want to build, the most fundamental invariant quantity is the interval

$$ds^2 = \eta_{ab} dy^a dy^b. \tag{40}$$

We now want to express the interval through quantities that depend on world indices and require its invariance under GCT. To this end, we introduce new fields $e^a_\mu(x)$ such that

$$dy^a = e^a_\mu(x) dx^\mu, \tag{41}$$

and rewrite (40) as follows,

$$ds^2 = \eta_{ab} e^a_\mu(x) e^b_\nu(x) dx^\mu dx^\nu \equiv g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{42}$$

Here $g_{\mu\nu}(x)$ is a new tensor field which is manifestly symmetric, and it is tempting to interpret it as a gauge field. Under GCT dx^μ transforms as follows,

$$dx'^\mu = J^\mu_\nu(x) dx^\nu, \qquad J^\mu_\nu(x) \equiv \frac{\partial x'^\mu}{\partial x^\nu}(x). \tag{43}$$

The invariance of (42) under GCT implies that $g_{\mu\nu}(x)$ must transform as

$$g'_{\alpha\beta} = (J^{-1})^\mu_\alpha g_{\mu\nu} (J^{-1})^\nu_\beta, \tag{44}$$

or, in short notation,

$$x' = Jx, \qquad e' = J^{-1} e, \qquad g' = (J^{-1})^T g J^{-1}. \tag{45}$$

(In the last expression, g should not be confused with the determinant of $g_{\mu\nu}$.) Eqs.(45) are analogous to those of transformations of YM–fields given by (29).

Note that, along with (45), the interval (42) is also invariant with respect to local Lorentz transformations

$$e'^a_\mu = \Lambda^a_c(x) e^c_\mu(x), \qquad \eta_{ab} \Lambda^a_c(x) \Lambda^b_d(x) = \eta_{cd}. \tag{46}$$

This is the vestage of the (global) Lorentz invariance of (40). At this step it seems that there is no need to gauge the Lorentz transformations since the current for them is not EMT but rather the angular–momentum tensor.

Define $g^{\mu\nu}$ such that $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. We can use the fields $g_{\mu\nu}$, $g^{\mu\nu}$ to rise and lower the world indices. For example,

$$x_\mu \equiv g_{\mu\nu}x^\nu, \qquad \partial^\mu = g^{\mu\nu}\partial_\nu, \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \qquad \partial_\mu x^\nu = \delta^\nu_\mu. \tag{47}$$

It follows that the quantities with upper indices transform with J matrix, while those with down indices transform with J^{-1} matrix. In particular,

$$g'^{\mu\nu} = J^\mu_\alpha J^\nu_\beta g^{\alpha\beta}. \tag{48}$$

The next step in building invariant action is to define an invariant measure. In special–relativistic field theories this is 4–volume $dV = d^4y$. Using (36), we write

$$d^4y = d^4x \left|\frac{\partial y}{\partial x}\right| = d^4x \det e^a_\mu. \tag{49}$$

Since

$$g \equiv \det g_{\mu\nu} = \det(e^a_\mu e^b_\nu \eta_{ab}) = -(\det e)^2, \tag{50}$$

it follows that

$$d^4y = \sqrt{-g}d^4x. \tag{51}$$

The r.h.s. of (51) is manifestly invariant under GCT.

1.3.2 Matter sector

Now we turn to covariantizing matter fields. As an example, consider the real massive scalar field ϕ. Its transformation properties under the global Poincare group are determined by

$$\phi'(y') = \phi(y). \tag{52}$$

The law (52) can be readily promoted to the transformation law under GCT:

$$\phi'(x') = \phi(x). \tag{53}$$

Then, the invariant action reads,

$$S_m = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right). \tag{54}$$

Its variation with respect to $g^{\mu\nu}$ gives,

$$\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{1}{2}\sqrt{-g}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - m^2\phi^2)\right). \tag{55}$$

We see that

$$\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}, \tag{56}$$

where $T_{\mu\nu}$ is obtained from (9) by promoting the Lorentz indices to the world ones, and replacing η_{ab} with $g_{\mu\nu}$. Thus, $T_{\mu\nu}$ is indeed the source of the gravitational force mediated by the field $g_{\mu\nu}$.

In deriving (55) we used the relations

$$\delta(g_{\mu\nu}g^{\nu\rho}) = \delta(\delta^\rho_\nu) = 0, \qquad \delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \tag{57}$$

and

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu} \tag{58}$$

following from

$$\delta \det M = \det(M+\delta M) - \det M = e^{\mathrm{Tr}\log(M+\delta M)} - e^{\mathrm{Tr}\log M} = \mathrm{Tr}(M^{-1}\delta M). \tag{59}$$

The equation of motion for the field ϕ reads as follows,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + m^2)\phi = 0. \tag{60}$$

As we will discuss later, this reduces to the Schrodinger equation in the limit of non–relativistic ϕ and weak gravitational fields.

1.3.3 Gravity sector

Let us now provide the field $g_{\mu\nu}$ with dynamics. Consider, for instance, the vector field $V^\mu(x)$ which transforms as

$$V'^\mu(x') = J^\mu_\nu(x)V^\nu(x). \tag{61}$$

Then, by the analogy with YM–theories, one should introduce the covariant derivatives D_μ whose transformations properties under GCT are

$$D_\mu V^\nu \to D'_\mu V'^\nu = (J^{-1})^\sigma_\mu J^\nu_\rho D_\sigma V^\rho. \tag{62}$$

For D_μ we write,

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho. \tag{63}$$

Then, Eq.(62) is valid as long as

$$\Gamma'^\lambda_{\mu\nu} = (J^{-1})^{\mu'}_\mu (J^{-1})^{\nu'}_\nu J^\lambda_{\lambda'}\left(\Gamma^{\lambda'}_{\mu'\nu'} + (J^{-1})^{\lambda'}_\sigma \partial_{\mu'} J^\sigma_{\nu'}\right). \tag{64}$$

The quantities $\Gamma^\lambda_{\mu\nu}$ can be expressed in terms of $g_{\mu\nu}$ as follows,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{65}$$

Determined by Eq.(65), $\Gamma^{\lambda}_{\mu\nu}$ are called Levi–Civita connection. The easiest way to derive Eq.(65) is to implement the metricity condition

$$D_{\alpha}g_{\mu\nu} = 0. \tag{66}$$

This condition is necessary for the EP to hold. Eq.(66) implies the vanishing of the connection in the absence of the gravitational force, in which case we must be able to recover the original Poincare–invariant theory. To get (65) from (66), one can take a half of the combination $D_{\alpha}g_{\mu\nu} - D_{\mu}g_{\nu\alpha} - D_{\nu}g_{\alpha\mu}$.

Knowing (65), one can define the action of D_{μ} on arbitrary tensors:

$$D_{\mu}T^{\alpha\beta\ldots}_{\rho\sigma\ldots} = \partial_{\mu}T^{\alpha\beta\ldots}_{\rho\sigma\ldots} + \Gamma^{\alpha}_{\mu\nu}T^{\nu\beta\ldots}_{\rho\sigma\ldots} + \ldots - \Gamma^{\nu}_{\mu\rho}T^{\alpha\beta\ldots}_{\nu\sigma\ldots} - \ldots\ . \tag{67}$$

Note that the connection does not transform as a tensor under GCT.

Proceeding as for YM–theories, we introduce the field strength tensor (using again the vector field as an example):

$$[D_{\mu}, D_{\nu}]V^{\beta} = R^{\beta}{}_{\mu\nu\alpha}V^{\alpha}. \tag{68}$$

This gives,

$$R^{\beta}{}_{\mu\nu\alpha} = \partial_{\mu}\Gamma^{\beta}{}_{\nu\alpha} - \partial_{\nu}\Gamma^{\beta}{}_{\mu\alpha} + \Gamma^{\beta}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\alpha} - \Gamma^{\beta}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\alpha}, \tag{69}$$

in close analogy with the YM field strength tensor. Using this tensor, one can define

$$R_{\nu\alpha} = R^{\mu}{}_{\mu\nu\alpha}, \qquad R = g^{\nu\alpha}R_{\nu\alpha}. \tag{70}$$

Note that the quantity R is invariant under CGT.

Which of $R^{\beta}{}_{\mu\nu\alpha}$, $R_{\nu\alpha}$, R should we put into S_g? To answer this question, consider the weak field approximation

$$g_{\mu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{71}$$

where κ is some small constant. Then we have, schematically,

$$\Gamma^{\lambda}_{\mu\nu} \sim \partial h, \qquad R^{\beta}_{\mu\nu\alpha},\ R_{\nu\alpha},\ R \sim (\partial^2 h, \partial h \partial h). \tag{72}$$

These expressions are different from those in YM–theories, where

$$F_{\mu\nu} \sim \partial A. \tag{73}$$

Moreover, the symmetry properties of YM field strength tensor and Riemann tensor $R^{\beta}{}_{\mu\nu\alpha}$ are different, and it is the latter that allows us to build a curvature scalar R, while $F_{\mu\nu}$ is manifestly antisymmetric. Hence, in case of gravity in the weak gravity limit the scalar curvature R is dominating, and we can write

$$S_{EH} = \int d^4x\sqrt{-g}\left(-\frac{2}{\kappa^2}R\right), \tag{74}$$

where $\kappa^2 = 32\pi G$, and the full invariant action is

$$S = S_{EH} + S_m. \tag{75}$$

Varying (75) with respect to $g^{\mu\nu}$ gives [7]

$$\delta S_{EH} = \int d^4x\sqrt{-g}\left(-\frac{2}{\kappa^2}R\right)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}, \tag{76}$$

$$\delta S_m = \int d^4x\sqrt{-g}\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}, \tag{77}$$

and the equations of motion are

$$\delta S = 0 \quad \Rightarrow \quad R_{\mu\nu} - \frac{1}{2}R = \frac{\kappa^2}{4}T_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{78}$$

This completes the construction of gravity as a gauge theory. Let us summarize our findings.

- We constructed GR by gauging ST translations,
- S_m gives the source of the gravitational field, namely, EMT, and
- S_{EH} gives the dynamics of the gravitational field.

2 Fermions in General Relativity

As was mentioned in Sec.1.3, covariantizing a (global) Poincare invariant theory may be a nontrivial task since it may not be possible to readily promote the Lorentz indices to the world ones. The example of this is the spinor field, the reason is fairly simple — the GCT group does not have spinor representations. Hence the procedure to embed fermions into curved space–time must be more laborious. Let $\mathcal{M}$ be a space–time manifold. At each point x of $\mathcal{M}$ one can consider the tangent space $T_x(\mathcal{M})$ which is Minkowskian. By joining $T_x(\mathcal{M})$ built at different x we obtain the tangent bundle over $\mathcal{M}$. It is natural to associate the spinor field with this tangent bundle. We then allow the global Lorentz transformations act independently at each layer of the bundle. In this way we obtain local Lorentz transformations in addition to GCT acting in $\mathcal{M}$. We then require the theory to be covariant under these two classes of transformations.

To make the theory GCT covariant, one should be able to relate the Lorentz coordinates of the original theory to the world coordinates of the curved manifold. In Sec.1.3 we introduced the objects that can make the

[7] If we impose non–trivial boundary conditions, the appropriate boundary term must be added to the action (74).

required relations. These are tetrads $e_a^\mu(x)$. In the geometric picture outlined above they naturally correspond to a change of a basis of vector fields in the tangent bundle from one coordinate frame to another,

$$\partial_a = e_a^\mu(x)\partial_\mu. \tag{79}$$

From Eq. (42) we see that

$$\eta_{ab} = e_a^\mu(x) e_b^\nu(x) g_{\mu\nu}(x). \tag{80}$$

Chosen in this way, the quantities $\{e_a^\mu(x)\}$ are said to form an orthonormal Vielbein basis. We observe the following properties of the tetrad fields [8],

$$e_{a\mu} e_b^\mu = \eta_{ab}, \quad e_{a\mu} e_\nu^a = g_{\mu\nu}. \tag{81}$$

Moreover, under local Lorentz transformations they transform as

$$e_b'^\mu(x) = \Lambda^a{}_b(x) e_a^\mu(x)\,. \tag{82}$$

From (79) it is clear that whenever one has an object with Lorentz indices, say, A_a, one can build an object with world indices A_μ by multiplying A_a by $e_\mu^a(x)$ [9].

We now require the local Lorentz covariance of the theory. This step is made in full analogy with the YM–theories discussed above. As an example, consider the theory of the Dirac field in flat four–dimensional space–time,

$$\mathcal{L} = \bar{\psi}(i\gamma^a\partial_a - m)\psi\,. \tag{83}$$

The field ψ transforms as

$$\psi \to S\psi \tag{84}$$

under (global) Lorentz transformations, where the matrix S has to satisfy the following conditions:

$$\begin{aligned} &\gamma_0 S^+ \gamma_0 = S^{-1}\,, \\ &S^{-1}\gamma^a \Lambda^b{}_a S = \gamma^b, \end{aligned} \tag{85}$$

where γ_a are four–dimensional Dirac matrices. The first condition above is dictated by the invariance of the mass term in (83), while the second — by that of the kinetic term.

The solution to Eqs. (85) is given by

$$S = \exp\Big\{ -\frac{iJ_{ab}\alpha^{ab}}{2}\Big\}\,, \tag{86}$$

[8] The Lorentz indices are raised and lowered with the metric η_{ab}.

[9] Note that this procedure can equally well work when covariantizing the usual tensor quantities. With the appropriate choice of connection, however, there would be no difference from the results obtained in Sec.1.

where $\alpha^{ab} = -\alpha^{ba}$ is the antisymmetric matrix of transformation parameters and J_{ab} are the generators of the Lorentz group in the spinorial representation,

$$J_{ab} = \frac{\sigma_{ab}}{2} = i\frac{[\gamma_a, \gamma_b]}{2} \,. \tag{87}$$

Notice that the spin generators defined in this way are antisymmetric and satisfy the usual commutation relations of the Lorentz algebra,

$$[J_{ab}, J_{cd}] = i\left[\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right] \,. \tag{88}$$

Now we upgrade the theory (83) of free fermions by gauging the Lorentz group. We require the invariance under the transformations (86), where α^{ba} are now functions of space–time coordinates. We introduce a gauge field A_μ and a covariant derivative,

$$\mathcal{D}_\mu\psi \equiv (\partial_\mu - igA_\mu)\psi \equiv (\partial_\mu - i\frac{g}{2}J_{ab}A_\mu^{ab})\psi \,. \tag{89}$$

The covariant derivative must transform homogeneously with respect to the gauge transformations,

$$(\mathcal{D}_\mu\psi)(x) \to S(x)(\mathcal{D}_\mu\psi)(x) \,, \tag{90}$$

which implies the following transformation law:

$$A'_\mu = SA_\mu S^{-1} - \frac{2i}{g}(\partial_\mu S)S^{-1} \,. \tag{91}$$

In what follows, we will call A_μ the “spin connection”.

Making use of tetrads and covariant derivatives we can rewrite the Lagrangian (83) in a covariant form,

$$\mathcal{L} = \bar{\psi}(i\gamma^a e_a^\mu(x)\mathcal{D}_\mu - m)\psi \,. \tag{92}$$

Note that the procedure outlined above can be generalized straightforwardly to general representations of the Lorentz group. In the general case the covariant derivative reads,

$$\mathcal{D}_\mu B_i = (\delta_i^j\partial_\mu - i\frac{g}{2}[J_{ab}^{(R)}]_i^j A_\mu^{ab})B_j \,, \tag{93}$$

where $[J_{ab}^{(R)}]_j^i$ are the generators of the Lorentz group in some representation. In this notation the infinitesimal transformations of the field B^i take the form

$$\delta B_i = -\frac{i}{2}[J_{ab}^{(R)}]_i^j\alpha^{ab}(x)B_j \,. \tag{94}$$

The next step in building the covariant theory is to define the field strength tensor,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -igJ_{ab}R_{\mu\nu}^{ab} \,, \tag{95}$$

where

$$R^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + g(A^a_{\mu c}A^{cb}_\nu - A^a_{\nu c}A^{cb}_\mu) \,. \tag{96}$$

The obvious candidate for the gauge field Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} R^{ab}_{\mu\nu} R^{\mu\nu}_{ab} \,. \tag{97}$$

If we studied usual non–abelian gauge theories, this would be the end of the story. In our case, however, we also have the tetrad field at hand. Using it we can construct new scalars for the Lagrangian density. For instance, we can contract both indices of the strength tensor with the tetrads to obtain [10]

$$R(A) = e^\mu_a e^\nu_b R^{ab}_{\mu\nu} \,. \tag{98}$$

We have now two ways to proceed. The first is to impose the first Vielbein postulate [11]

$$\mathcal{D}_\mu e^a_\nu = 0 = \partial_\mu e^a_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha - g[A_\mu]^{ac}\eta_{cb} e^b_\nu \,, \tag{99}$$

where we used the generators of the Lorentz group in the vector representation,

$$[J^{(V)}_{ab}]^i_j = -i(\delta^i_a \eta_{bj} - \delta^i_b \eta_{aj}) \,. \tag{100}$$

Eq. (99) allows to relate the spin and world connections,

$$g A^{ab}_\mu = e^{\nu a}(\partial_\mu e^b_\nu - \Gamma^\alpha_{\mu\nu} e^b_\alpha) = e^{\nu a} D_\mu e^b_\nu. \tag{101}$$

Notice that the above relation can be used to uniquely define the Levi-Civita connection on $\mathcal{M}$. Had we used a more general connection, this condition would not completely fix it. In this case,

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_b \partial_\mu e^b_\nu - g A^{ab}_\mu e_{a\nu} e^\alpha_b \equiv \Gamma^{\alpha\,(LC)}_{\mu\nu} - g\tilde{A}^{ab}_\mu e_{a\nu} e^\alpha_b \,, \tag{102}$$

where $\Gamma^{\alpha\,(LC)}_{\mu\nu}$ is the symmetric Levi-Civita connection and $\tilde{A}^{ab}_\mu$ is an arbitrary function. Then one can define the Riemann tensor,

$$R_{\mu\nu\lambda\rho} = e_{a\lambda} e_{b\rho} R^{ab}_{\mu\nu} \,, \tag{103}$$

and the rest of GR follows.

The second way is go is to write down the following action,

$$S = \text{const} \times \int d^4x \sqrt{-g}\; e^\mu_a e^\nu_b R^{ab}_{\mu\nu} \,. \tag{104}$$

[10] Note that the constant g can be absorbed into the normalization of the gauge field A.

[11] It can be motivated by the requirement to be able to convert the Lorentz into world indices inside the total covariant derivative, e.g., $e^a_\mu \nabla_\nu V^\mu = \mathcal{D}_\nu V^a$.

Let us vary this action with respect to the spin connection. One has,

$$
\begin{aligned}
&\delta S = \text{const} \times \int d^4x\sqrt{-g}\; e^\mu_a e^\nu_b \delta R^{ab}_{\mu\nu} = \text{const} \times \int d^4x\sqrt{-g}\; e^\mu_a e^\nu_b \mathcal{D}_\mu \delta A^{ab}_\nu\,,\\
&\Rightarrow \quad \mathcal{D}_\mu e^\nu_a = 0\,,
\end{aligned}
\tag{105}
$$

so we again arrived at Eq. (99). The choice of the action (104) seems to be a big simplification though. We can get the same results considering a more generic action,

$$
S = \int d^4x\sqrt{-g}[aR(g) + bR(A) + c\mathcal{D}_\mu e^b_\nu \mathcal{D}^\nu e^\mu_b - \Lambda]\,. \tag{106}
$$

The higher order terms, however, are forbidden since their variation with respect to the spin connection will not yield a constraint like (105), but rather a dynamical equation.

3 Weak field gravity

Let us study some basic features of gravity in the weak field limit. This amounts to expanding the metric around the Minkowski background,

$$
g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{107}
$$

where $\kappa^2 \equiv 32\pi G$. Then the Ricci tensor and the Ricci scalar read,

$$
\begin{aligned}
R_{\mu\nu} &= \frac{\kappa}{2}\left[\partial_\mu\partial_\lambda h^\lambda_\nu + \partial_\nu\partial_\lambda h^\lambda_\nu - \partial_\mu\partial_\nu h^\lambda_\lambda - \Box h_{\mu\nu}\right] + O(h^2)\,,\\
R &= \kappa\left[\partial_\mu\partial_\lambda h^{\mu\lambda} - \Box h^\lambda_\lambda\right] + O(h^2)\,.
\end{aligned}
\tag{108}
$$

The Einstein equation reads,

$$
R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \equiv \frac{\kappa}{2}O_{\mu\nu\alpha\beta}h^{\alpha\beta} = \frac{\kappa^2}{4}T_{\mu\nu}\,. \tag{109}
$$

It is convenient to introduce the following “identity” tensor,

$$
I_{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})\,, \tag{110}
$$

making use of which the equation defining the Green function of Eq. (109) can be written as,

$$
O_{\mu\nu}{}^{\alpha\beta}G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2}I_{\mu\nu\gamma\delta}\delta^{(4)}_D(x-y)\,, \tag{111}
$$

where,

$$
O^{\mu\nu}{}_{\alpha\beta} \equiv (\delta^{(\mu}_\alpha\delta^{\nu)}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta})\Box - 2\delta^{(\mu}_{(\alpha}\partial^{\nu)}\partial_{\beta)} + \eta_{\alpha\beta}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial_\alpha\partial_\beta\,. \tag{112}
$$

As usual in gauge field theories, the operator $O_{\mu\nu}{}^{\alpha\beta}$ cannot be inverted. For that one has to do gauge–fixing.

3.1 Gauge transformations

Consider an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \kappa \xi^{\mu}(x)\,. \tag{113}$$

Then, the transformed metric $h'_{\mu\nu}$ takes the following form

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\,. \tag{114}$$

Note that the curvature invariants do not transform, e.g. $R' = R$. Now let's choose a gauge. A particularly convenient is the de Donder (harmonic) gauge, defined as,

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda = 0\,. \tag{115}$$

In order to go to this gauge one has to choose $\xi^\mu : \Box \xi_\mu = -(\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda)$. One can introduce the field

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_\lambda\,, \tag{116}$$

using which Eq. (109) can be rewritten as,

$$\Box \bar{h}_{\mu\nu} = -\frac{\kappa}{2} T_{\mu\nu}\,. \tag{117}$$

The Green function of Eq. (111) then reads,

$$G_{\mu\nu\alpha\beta} = \frac{1}{2\Box}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)\delta^{(4)}_D(x-y)\,. \tag{118}$$

3.2 Newton's law

Eq. (109) can also be rewritten as,

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda_\lambda)\,. \tag{119}$$

For a point source with $T_{00} = M\delta^{(3)}(\mathbf{x}), T_{ij} = 0$,

$$T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda_\lambda = \frac{1}{2} M \delta^{(3)}(\mathbf{x}) \times \mathrm{diag}(1,1,1,1)\,. \tag{120}$$

Plugging an ansatz $\kappa h_{\mu\nu} = 2\Phi_g \mathrm{diag}(1,1,1,1)$ we obtain the solution,

$$\Phi_g = -\frac{\kappa^2 M}{32\pi}\frac{1}{r} = -\frac{GM}{r}\,. \tag{121}$$

3.3 Gauge invariance for a scalar field

Take a look at the Lagrangian for a free minimally coupled scalar field,

$$\mathcal{L} = \frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right] . \tag{122}$$

For small gauge transformations one has,

$$\begin{aligned} g'^{\mu\nu} &= g^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu , \\ \partial'_\mu &= \partial_\mu - (\partial_\mu\xi^\nu)\partial_\nu , \\ \phi'(x') &= \phi(x) . \end{aligned} \tag{123}$$

Then it is straightforward to obtain that the Lagrangian does not change under the gauge transformations.

3.4 Schrödinger equation

Let's look at the Klein-Gordon-Fock equation,

$$(\Box + m^2)\phi = 0 \tag{124}$$

In the harmonic coordinates the d'Alembertian reads,

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) = g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu})\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu , \tag{125}$$

where in the last equality we made use of Eq. (107) and the definition of the harmonic gauge,

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = -\kappa\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\lambda_\lambda + O(h^2)\right) \simeq 0 . \tag{126}$$

We will use the metric for a static external gravitational field,

$$g_{00} = 1 - 2\Phi_g , \quad g_{ij} = -(1 + 2\Phi_g)\delta_{ij} , \quad \Phi_g \ll 1 . \tag{127}$$

Let's perform a non-relativistic reduction for the wavefunction of the filed ϕ,

$$\phi = e^{-imt}\psi(t, \mathbf{x}) \tag{128}$$

Plugging this into Eq. (124) we find,

$$[(1 + 2\Phi_g)(-m^2 - 2im\partial_0 + \partial_0^2) - \delta^{ij}\partial_i\partial_j + m^2]\psi(t, \mathbf{x}) = 0 . \tag{129}$$

One observes that the mass term cancels to the leading order in Φ_g, and we are left with the usual Schrödinger equation for a particle in an external gravitational field,

$$i\partial_0\psi = \left[-\frac{\Delta}{2m} + m\Phi_g\right]\psi . \tag{130}$$

Note that one can consistently compute corrections to the Schrödinger equation. For instance, the Lagrangian for a two body system (so-called Einstein-Infeld-Hoffman Lagrangian) reads [4]

$$\begin{aligned}\mathcal{L} =& \frac{m_1 \mathbf{v}_1^2}{2} + \frac{m_2 \mathbf{v}_2^2}{2} + \frac{m_1 \mathbf{v}_1^4}{8} + \frac{m_2 \mathbf{v}_2^4}{8} \\ &+ \frac{G m_1 m_2}{2r} \left[3(\mathbf{v}_1 + \mathbf{v}_2)^2 - 7(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{r^2} - \frac{G(m_1 + m_2)}{r} \right] .\end{aligned} \tag{131}$$

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