Lecture 4 — Supplement

13 October 2016

1 Infrared properties of General Relativity

Early work on the quantum field theory of general relativity focused on its UV properties. For example, we discussed in detail divergences arising from loops in pure gravity and with matter. More recently, the effective field theory approach showed how to obtain quantum predictions at low energy. Here we want to explore the lowest energy limit and describe the IR structure of GR. Early developments in this field go back to works by Weinberg [1], Jackiw [2], Gross and Jackiw [3]. However, the most intriguing results, as well as new insights into the old studies, have been obtained very recently, with the development of new powerful techniques allowing to handle the complicated structure of gravity amplitudes. Below we will briefly describe some of these classical and new results, focusing mainly on pure gravity in four-dimensional space-time.

1.1 IR divergences at one loop

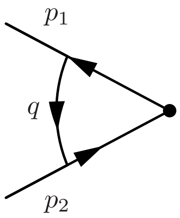
We start with the discussion of IR divergences in one-loop diagrams. As an example, consider the graviton-graviton scattering process. The amplitude of this process depends on helicities of incoming and outgoing particles. At tree level, summing up all diagrams contributing to the scattering, we have [4]

$$i\mathcal{M}_{tree}(++;++) = \frac{i}{4}\kappa^2\frac{s^3}{tu}\,, \quad i\mathcal{M}_{tree}(-+;-+) = -\frac{i}{4}\kappa^2\frac{u^3}{st}\,, \tag{1}$$

$$i\mathcal{M}_{tree}(++;+-) = i\mathcal{M}_{tree}(++;--) = 0\,. \tag{2}$$

In these expressions, the first pair of signs in $\mathcal{M}_{tree}$ denotes the helicities of incoming gravitons, and the second pair — those of outgoing gravitons.

To go to one loop, we insert a virtual graviton propagator into the tree diagrams, in all possible ways. Not all diagrams, obtained in this way, give rise to IR divergences. To illustrate this point, consider the scattering of massless scalar particles at one loop. In four dimensions, the measure of the loop integral, $d^4q \sim |q|^3 d|q|$, suppresses the soft divergence unless at least three adjacent propagators vanish simultaneously. Indeed, in the latter case,

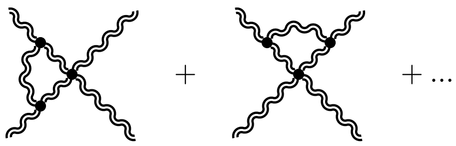


$$\sim \int d^4q \frac{1}{(p_1+q)^2 q^2 (p_2+q)^2} \,, \tag{3}$$

and this expression diverges in the limit $q \to 0$ provided that $p_1^2 = p_2^2 = 0$. To see this, one evaluates the integral in dimensional regularization, which gives

$$\frac{-ir_\Gamma}{(4\pi)^{2-\epsilon}(-(p_1+p_2)^2)^{1+\epsilon}} \frac{1}{\epsilon^2} \,, \tag{4}$$

where $r_\Gamma = \Gamma^2(1-\epsilon)\Gamma(1+\epsilon)/\Gamma(2-\epsilon)$. Going back to the four-graviton scattering, we conclude that one-loop diagrams, in which both ends of the virtual graviton propagator are attached to the same external line, do not contribute to the IR divergent part of the amplitude. Hence, to capture the IR divergence, it is enough to consider the diagrams of the form



(5)

Let us look at the specific helicity configuration $(-+;-+)$. Summing over all pairs of lines, to which the internal propagator is attached, one gets the expected IR divergence [5]

$$ir_\Gamma \frac{\kappa^2}{(4\pi)^{2-\epsilon}} \left(\frac{s\log(-s) + t\log(-t) + u\log(-u)}{2\epsilon} \right) \times \mathcal{M}_{\textit{tree}}(-+;-+) \,. \tag{6}$$

This reproduces the full structure of divergences of the corresponding one-loop amplitude [6],

$$\begin{aligned} \mathcal{M}_{\textit{1-loop}}(-+;-+) &= ir_\Gamma \frac{stu\kappa^2}{4(4\pi)^{2-\epsilon}} \mathcal{M}_{\textit{tree}}(-+;-+) \\ &\times \left(\frac{2}{\epsilon} \left(\frac{\log(-u)}{st} + \frac{\log(-t)}{su} + \frac{\log(-s)}{tu} \right) + \text{finite terms} \right) \,, \end{aligned} \tag{7}$$

since in pure gravity at one loop there are no divergences from the UV regime. A similar divergence is present in $\mathcal{M}_{\textit{1-loop}}(++;++)$. The amplitudes with other helicity configurations contain no infinities.

Adding matter does not change qualitatively the soft behavior of one-loop amplitudes. For example, massless scalar-graviton scattering amplitudes experience the same kind of IR divergences from the virtual graviton propagator. Note, however, that the scalar loops do not contribute to soft infinities [4]. Hence, IR structure of gravity amplitudes is universal. Knowing this structure helps understand other properties of these amplitudes. For example, using the unitarity method outlined above, one can extract the information about infinities present in the amplitude. However, this method does not distinguish between IR and UV infinities. Knowing the form of IR divergences, therefore, allows to identify the remaining UV ones [5].

1.2 Cancellation of IR divergences

As we have just seen, some of one-loop gravity amplitudes contain IR divergences from virtual gravitons. Going to higher loops makes these divergences worse. However, there is another source of divergences, coming from the diagrams, where soft gravitons are radiated away from the hard particle lines. In general, such diagrams must be taken into account when computing any scattering process, since there is no possibility to distinguish experimentally the process, in which an arbitrary soft zero-charge particle is emitted, from the process without such particle. Hence, the question arises about the possible cancellation of IR divergences arising in diagrams with virtual and real soft gravitons. As was shown by Weinberg in [1], such cancellation indeed occurs, order by order in perturbation theory, in close analogy with QED. Consider, for example, some process involving hard scalar particles, and let the rate of this process without real or virtual soft gravitons taken into account be Γ_0. Then, including the possibility to emit soft gravitons with energies below some threshold E modifies Γ_0 as follows [1],

$$\Gamma(E) = \left(\frac{E}{\Lambda}\right)^B b(B)\Gamma_0 \,, \tag{8}$$

where

$$B = \frac{\kappa^2}{64\pi^2} \sum_{i,j} \eta_i \eta_j m_i m_j \frac{1+\beta_{ij}^2}{\beta_{ij}(1-\beta_{ij}^2)^{1/2}} \log\left(\frac{1+\beta_{ij}}{1-\beta_{ij}}\right) \,, \tag{9}$$

$$b(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{\sin y}{y} e^{x\int_0^1 \frac{d\omega}{\omega}(e^{i\omega y}-1)} \simeq 1 - \frac{\pi^2 x^2}{12} + \ldots \,, \tag{10}$$

the scale Λ defines our notion of "infrared", β_{ij} is the relative velocity of ith and jth particles,

$$\beta_{ij} = \left(1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}\right)^{1/2} \,, \tag{11}$$

m_i and p_i are the ith particle mass and momentum, and

$$\eta_i = \begin{cases} -1 & \text{for incoming } i\text{th particle,} \\ +1 & \text{for outgoing } i\text{th particle.} \end{cases} \tag{12}$$

The expression (8) is, in fact, universal in the sense that its form does not depend on masses and spins of hard particles. In particular, in remains valid if some of the masses m_i vanish, since an apparent singularity in B in this limit is removed due to momentum conservation. This fact makes gravity different from QED, where the charged massless hard particles do lead to additional divergences.

The proof of cancellation of IR divergences is based on a crucial observation that diagrams in which a soft real or virtual graviton line is attached to another soft real graviton line do not contribute to the divergent part of the amplitude. Indeed, the effective coupling for the emission of a soft graviton from another soft graviton of energy E is proportional to E, and the vanishing of this coupling prevents simultaneous IR divergence from the one graviton line attached to another. We observe another difference from the case of QED, where such diagrams are forbidden due to the electrical neutrality of the photon.

Let us go back to the four-graviton scattering process studied previously. After taking into account both radiative and one-loop corrections to the tree-level amplitude, the answer becomes finite. For example, for the differential cross-section we have [7]

$$\begin{aligned}
&\left(\frac{d\sigma}{d\Omega}\right)_{tree} + \left(\frac{d\sigma}{d\Omega}\right)_{rad.} + \left(\frac{d\sigma}{d\Omega}\right)_{nonrad.} = \\
&\frac{\kappa^4 s^5}{2048\pi^2 t^2 u^2}\left\{1 + \frac{\kappa^2 s}{16\pi^2}\left[\log\frac{-t}{s}\log\frac{-u}{s} + \frac{tu}{2s^2} f\left(\frac{-t}{s}, \frac{-u}{s}\right)\right.\right. \\
&\left.\left. -\left(\frac{t}{s}\log\frac{-t}{s} + \frac{u}{s}\log\frac{-u}{s}\right)\left(2\log(2\pi^2) + \gamma + \log\frac{s}{\Lambda^2} + \frac{\sum_{ij}\eta_i\eta_j\mathcal{F}^{(1)}(\gamma_{ij})}{\sum_{ij}\eta_i\eta_j\mathcal{F}^{(0)}(\gamma_{ij})}\right)\right]\right\},
\end{aligned} \tag{13}$$

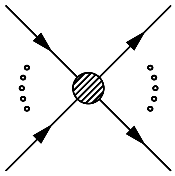
where

$$\begin{aligned}
f\left(\frac{-t}{s}, \frac{-u}{s}\right) =& \frac{(t+2u)(2t+u)(2t^4 + 2t^3 u - t^2 u^2 + 2tu^3 + 2u^4)}{s^6}\left(\log^2\frac{t}{u} + \pi^2\right) \\
&+ \frac{(t-u)(341t^4 + 1609t^3 u + 2566t^2 u^2 + 1609tu^3 + 341u^4)}{30s^5}\log\frac{t}{u} \\
&+ \frac{1922t^4 + 9143t^3 u + 14622t^2 u^2 + 9143tu^3 + 1922u^4}{180s^4}.
\end{aligned} \tag{14}$$

1.3 Weinberg's soft theorem and BMS transformations

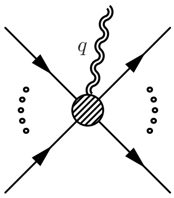
The modification of an on-shell diagram obatined by attaching a soft real graviton line to some external hard line leads to the appearance of an additional pole in the amplitude corresponding to this diagram. It turns out that in general the contribution from this pole can be separated from the rest of the amplitude, and that the amplitude of some process with one real soft graviton is given by the amplitude of the process without such graviton times a universal "soft factor". This is essentially the statement of the soft theorem proven by Weinberg in [1]. As an illustrative example, consider the on-shell diagram whose external lines are massless scalar particles with momenta p_i, $i = 1, ..., n$,

$$i\mathcal{M}(p_1, ..., p_n) = \quad . \tag{15}$$

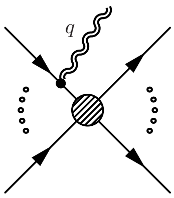


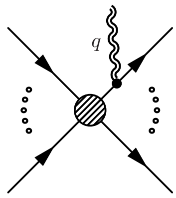
Now we want to attach an outgoing soft graviton with momentum q to this diagram in all possible ways. The dominant contribution to the modified amplitude in the limit $q \to 0$ is then given by

$$i\mathcal{M}_{\mu\nu}(p_1, ..., p_n, q) =$$



$$= \sum \qquad + \sum \qquad . \tag{16}$$





Note that the diagrams with the external graviton attached to internal lines do not contribute to the soft pole. The leading term of the expansion of $i\mathcal{M}_{\mu\nu}$ around $q = 0$ is written as follows [1],

$$i\mathcal{M}_{\mu\nu}(p_1, ..., p_n, q) = \frac{i\kappa}{2} \sum_{i=1}^{n} \frac{\eta_i p_{i\mu} p_{i\nu}}{p_i \cdot q} \mathcal{M}(p_1, ..., p_n) \,, \tag{17}$$

where η_i is defined in (12). The soft factor, that gives a pole in this expression, is universal in the sense that it does not depend on the spins of hard particles. A similar soft theorem is known to hold for color-ordered amplitudes in the YM-theory.

Eq.(17) relates two amplitudes to the leading order in the soft graviton energy. It can be verified straightforwardly without much effort, though its generalization to other types of hard particles is not obvious. Recently, a new way of thinking about such relations appeared. Whenever one has a statement about soft behavior of the theory, it is tempting to work out some symmetry arguments which lead to a desired consequence in the low-energy limit. We already saw one nice example of this situation, when we studied the low-energy behavior of the four-pion scattering amplitude in the linear sigma-model. The vanishing of the amplitude at zero momentum transfer is, in fact, a consequence of the degeneracy of vacua of the theory. Hence, it is natural to assume that the Weinberg's soft theorem (17) can also be viewed as a consequence of some symmetry the quantum gravity S-matrix obeys [1]. This line of research was taken in [8],[9], where such symmetry was identified as an "anti-diagonal" subgroup of $\mathrm{BMS}^{+}\times\mathrm{BMS}^{-}$ transformations [2]. Let us describe briefly what these transformations are.

Describing scattering processes in quantum gravity, we restrict ourselves with asymptotically flat space-time geometries, in which case Minkowski space-time can be taken as both in and out vacuum state. The properties of asymptotically flat space-times are well-known. To study their behavior at future I^{+} and past I_{-} null infinities, it is convenient to use, correspondingly, retarded and advanced Bondi coordinates. Near I^{+}, the metric can be written as [12]

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} + ... \end{aligned} \tag{18}$$

Here $u = t - r$ is the retarded time, $\gamma_{z\bar{z}} = 2(1+z\bar{z})^{-2}$ is the metric of a unit sphere, C_{zz}, $C_{\bar{z}\bar{z}}$ are functions of $u, z, \bar{z}$; $U_z = -\frac{1}{2}D^z C_{zz}$, where the covariant derivative D^z is defined with the metric $\gamma_{z\bar{z}}$, and dots mean the higher-order terms in $1/r$-expansion. All future asymptotic data are encoded by the Bondi mass aspect $m_B = m_B(u, z, \bar{z})$, determining local energy at retarded time u and at a given angle $(z, \bar{z})$, and by the Bondi news $N_{zz} = \partial_u C_{zz}$ determining the outgoing flux of radiation. Similarly, near I_{-} the metric takes the form

$$\begin{aligned} ds^2 &= -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 - 2V_z dvdz - 2V_{\bar{z}}dvd\bar{z} + ... \end{aligned} \tag{19}$$

where $V_z = \frac{1}{2}D^z D_{zz}$, and the corresponding Bondi news is $M_{zz} = \partial_v D_{zz}$. Eqs.(19) and (18) can be considered as initial and final data for the gravitational scattering process. To represent a valid solution to the scattering

[1] The soft graviton in this picture acquires a natural interpretation of a Nambu-Goldstone boson, associated with the spontaneous breakdown of the symmetry by the initial and final scattering data.

[2] The BMS transformations were first studied in [10],[11] in the context of gravitational waves.

problem, the initial data (m_B^-, M_{zz}) must, of course, be suitably related to the final data (m_B, N_{zz}).

One can Define BMS$^+$ transformations as a subgroup of the diffeomorphisms that acts non-trivially on the future asymptotic data (m_B, N_{zz}). Similarly, define BMS$^-$ transformations as consisting of those diffeomorphisms that act non-trivially on the past asymptotic data (m_B^-, M_{zz}). Besides the usual Poincare group, BMS$^\pm$ include also an infinite-dimensional class of "large" diffeomorphisms called supertranslations. They generate arbitrary angle dependent translations of retarded (advanced) time variable.

Consider now some scattering process, and let (m_B^-, M_{zz}) and (m_B, N_{zz}) be the initial and final data correspondingly (representing, e.g., the pulses of gravitational radiation). A BMS$^-$ transformation maps the initial state into another state $(\tilde{m}_B^-, \tilde{M}_{zz})$. One can argue that there is always a transformation from BMS$^+$ that maps the final state onto $(\tilde{m}_B, \tilde{N}_{zz})$ in such a way that $\langle m_B, N_{zz}|S|m_B^-, M_{zz}\rangle = \langle \tilde{m}_B, \tilde{N}_{zz}|S|\tilde{m}_B^-, \tilde{M}_{zz}\rangle$. And vice versa, given a BMS$^+$ transformation, one can find the one from BMS$^-$ to keep the matrix element unchanged. This means that the quantum gravity S-matrix commutes with the infinite sequence of generators of the subgroup BMS^0 of BMS$^+\times$BMS$^-$. In turn, this implies the existence of Ward identities associated to the BMS^0-symmetry. As was shown in [9], these Ward identities lead to the Weinberg's soft theorem (17). And vice versa, from the expression (17) one can deduce the Ward identities associated with some symmetry of the S-matrix, the symmetry group being BMS^0.

The symmetry arguments outlined above make manifest the universal nature of the soft theorem: the soft-graviton limit of any gravitational scattering amplitude at the leading order in a soft momentum is given by Eq. (17).

1.4 Other soft theorems

Here we outline various generalizations of the Weinberg's soft theorem and its counterparts in YM-theories, that are discussed in recent literature. For convenience, we omit the coupling constant κ in the gravity amplitudes, and absorb the factors η_i into the momenta of hard particles.

1.4.1 Cachazo-Strominger soft theorem

One natural way to generalize the expression (17) is to extend it by including sub-leading terms in the soft momentum expansion. For tree-level gravitational single-soft graviton amplitudes the extended soft theorem takes the form [13]

$$i\epsilon^{\mu\nu}\mathcal{M}_{\mu\nu}(p_1, ..., p_n, q) = (S^{(0)} + S^{(1)} + S^{(2)})i\mathcal{M}(p_1, ..., p_n) + \mathcal{O}(q^2)\,, \quad (20)$$

where $\epsilon_{\mu\nu}$ is the soft graviton polarization tensor obeying $\epsilon_{\mu\nu}q^\nu = 0$. In Eq.(20), the term $S^{(0)}$ is the Weinberg's leading-order universal soft factor, that we have already discussed,

$$S^{(0)} = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot q} \,. \tag{21}$$

Note again that the form of $S^{(0)}$ can be deduced from symmetry considerations, namely, from the expected invariance of the S-matrix with respect to supertranslations. The term $S^{(1)}$ provides a sub-leading correction to the Weinberg's theorem,

$$S^{(1)} = -i \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} p_i^\mu (q_\rho J_i^{\rho\nu})}{p_i \cdot q} \,, \tag{22}$$

with $J_i^{\rho\nu}$ the total angular momentum of the ith hard particle. It was argued that the term (22) can also be obtained from symmetry considerations, and the corresponding S-matrix symmetry is the extension of BMS transformations obtained by including all Virasoro transformations ("superrotations") of the conformal sphere. Finally, the $S^{(2)}$ term is found to be

$$S^{(2)} = -\frac{1}{2} \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} (q_\rho J_i^{\rho\mu})(q_\sigma J_i^{\sigma\nu})}{p_i \cdot q} \,. \tag{23}$$

The origin of this term from symmetry arguments is also discussed in the literature [14]. Let us comment on Eq. (20).

- It was proven to hold for all graviton tree-level amplitudes with one real soft graviton. Hence the terms $S^{(j)}$ are universal, at least at tree level.
- The gauge invariance requires that the pole terms vanish for $\delta_\Lambda \epsilon_{\mu\nu} = \Lambda_\mu q_\nu + \Lambda_\mu q_\nu$ with $\Lambda \cdot q = 0$. Indeed, $\delta_\Lambda S^{(0)} = 0$ due to global energy-momentum conservation, $\delta_\Lambda S^{(1)} = 0$ due to global angular momentum conservation, and $\delta_\Lambda S^{(2)} = 0$ because $J_i^{\mu\nu}$ is antisymmetric.
- When taking the soft limit $q \to 0$, the momenta of some hard particles must be deformed because of the momentum conservation, and this deformation is ambiguous. Hence the expansion about the soft limit is not unique. The expression (20) holds for a very large class of such soft limit expansions. It remains to be verified if it holds for every conceivable definition of the soft limit expansion.

1.4.2 1-loop corrections to Cachazo-Strominger soft theorem

If one takes for granted that the theorem (20) is deduced from the symmetry arguments, a natural question is whether it is an exact statement in perturbation theory. Naively, one would expect loop corrections to the sub-leading soft factors. Indeed, due to the dimensionful coupling in gravity, the dimensional analysis requires loop corrections to be suppressed by extra powers of soft invariants. As a result, $S^{(0)}$ must be exact to all orders. In [15],[16] the one-loop corrections to the sub-leading factors were studied for particular helicity configurations. It was shown that for "all-plus" amplitudes the terms $S^{(1)}$ and $S^{(2)}$ receive no corrections from one loop. The same is true for "single-minus" amplitudes with the negative helicity of the soft graviton, while for the "single-minus" amplitudes with the positive helicity soft graviton the term $S^{(2)}$ does require loop corrections.

1.4.3 Relation to YM-theories

As we showed in these Lectures, GR has many common properties with other gauge theories. It is, therefore, natural to expect the analogs of the soft theorems described above to hold in YM-theories. The expectation is completely justified by the recent study of the soft behaviour of YM-amplitudes. In particular, an analysis of color-ordered tree-level amplitudes including soft gluon reveals universal soft behavior of the form [17]

$$\mathcal{M}_{YM}(p_1, ..., p_n, q) = (S^{(0)}_{YM} + S^{(1)}_{YM})M(p_1, ..., p_n) + \mathcal{O}(q)\,, \tag{24}$$

where $S^{(0)}_{YM}$ and $S^{(1)}_{YM}$ are leading and sub-leading universal soft factors analogous to (21) and (22). The term $S^{(0)}_{YM}$ can be understood through the symmetry arguments similar to those in the case of GR [18]. As for $S^{(1)}_{YM}$, no such arguments are known yet. Contrary to the case of GR, both $S^{(0)}_{YM}$ and $S^{(1)}_{YM}$ receive corrections at one-loop level for amplitudes with particular helicity configurations [16].

Before we explored one nice example of a deep connection between GR and YM-theories, by discussing how gravity amplitudes can be derived from the corresponding YM-amplitudes, via KLT-relation. One can expect that the soft limit of gravity amplitudes can also be deduced from that of YM-amplitudes. As was shown in [19], the leading and sub-leading soft factors in GR can indeed be reproduced by the leading and sub-leading soft factors of YM-amplitudes. Schematically,

$$S^{(0)} + S^{(1)} + S^{(2)} \sim \left(S^{(0)}_{YM} + S^{(1)}_{YM}\right)^2 . \tag{25}$$

This expression is one more example of how apparently different theories are related to each other in a deep and beautiful way.

Finally, we note that similar soft theorems exist for supersymmetric extensions of GR and YM-theories, as well as for theories beyond four dimensions.

1.4.4 Double-soft limits of gravitational amplitudes

One more natural generalization of the soft theorems (17) and (20) is to consider the amplitudes with two or more soft gravitons. This direction of studies was recently carried out in [20],[21]. The very notion of the double-soft limit is ambiguous as it can be taken in two ways. Either one can send both graviton momenta q_1 and q_2 to zero uniformly, with $q_1/q_2 = \text{const}$, or one can take the consecutive limit q_1(or q_2)$\to 0$ after q_2(or q_1)$\to 0$. Both ways reveal the factorization property of the double-soft amplitudes, but, in general, with different universal soft factors. It is clear that in the case of the soft limit taken consecutively, the leading soft factor $S_2^{(0)}$ is given by the product of two single-soft-graviton factors. Namely, if we write (21) as $S^{(0)} = \sum_i \mathcal{S}_i^{(0)}$, where $i = 1, ..., n$ enumerates hard particle lines, then

$$\mathcal{M}(p_1, ..., p_n, q_{n+1}, q_{n+2}) \sim \sum_{i,j} \mathcal{S}_i^{(0)}(q_{n+1})\mathcal{S}_j^{(0)}(q_{n+2})\mathcal{M}(p_1, ..., p_n)\,. \qquad (26)$$

The statement remains valid for consecutive limits of any multi-soft amplitudes.

In [20], the leading and sub-leading soft factors were investigated at tree level for different helicity configurations of the soft gravitons. It was found that the leading factor $S_2^{(0)}$ does not depend on the way one takes the soft limit, regardless the relative polarizations of the gravitons. Hence, Eq. (26) expresses the universal double-soft behavior at the leading order in soft momenta. The sub-leading factor $S_2^{(1)}$, on the other hand, shows such dependence in the case when the polarizations are different. It is yet to be seen if the double- and multi-soft theorems can be thought of as originated from some symmetries of the quantum gravity S-matrix.

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