Note Title

EPFL Lectures

Quantization

1) Some basic facts – (weak field, Schrod. eq.)
2) Gravitons (Feynman rules)
3) Simple quantization
4) Background field method
5) Gauge fixing + ghost
6) Heat kernel methods
7) One loop effect

Basics - weak field gravity

$\kappa^2 = 32\pi G$

Weak field expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

Then

$$R_{\mu\nu} = \frac{\kappa}{2}\left[\partial_\mu \partial_\lambda h^\lambda{}_\nu + \partial_\nu \partial_\lambda h^\lambda{}_\nu - \partial_\mu \partial_\nu h^\lambda{}_\lambda - \Box h_{\mu\nu}\right] + \cdots$$

and

$$R = \kappa\left[\partial_\lambda \partial_\sigma h^{\lambda\sigma} - \Box h^\lambda{}_\lambda\right] + \cdots$$

We have

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = \frac{\kappa^2}{4} T_{\mu\nu}$$

Exercise: check $\partial_\mu\left[R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R\right] = 0$ at this order

Try to solve $\swarrow [\partial\partial \cdots]$

$$\underline{\mathcal{O}_{\mu\nu\alpha\beta}} \, h^{\alpha\beta} = \frac{K}{2} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{K^2}{4} T_{\mu\nu}$$

$\nwarrow \partial\partial h$

Greens function technique fails

$\swarrow I_{\mu\nu\gamma\delta}$

$$\mathcal{O}_{\mu\nu}{}^{\alpha\beta} \, G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2}\left(\eta_{\mu\gamma}\eta_{\nu\delta} + \eta_{\mu\delta}\eta_{\nu\gamma}\right)\delta^4(x-y) \quad \text{**}$$

$$\Longrightarrow \quad h^{(x)}_{\alpha\beta} = \int d^4y \, G_{\alpha\beta\gamma\delta}(x-y) \, \frac{K}{2} T_{\mu\nu}(y) \quad \text{**}$$

But $G_{\alpha\beta\gamma\delta}$ does not exist **

$\Longrightarrow$ need gauge fixing

Gauge invariance

Small coordinate change

$$x'^{\mu} = x^{\mu} + \underline{\kappa\, \xi^{\mu}(x)} \quad , \quad dx'^{\mu} = dx^{\mu} + \underline{\kappa\, \partial_\nu \xi^{\mu}}\, dx^{\nu}$$

Then with $g'_{\mu\nu} = \eta_{\mu\nu} + \kappa h'_{\mu\nu}$, we have

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Curvature are all

$R' = R$

We can use this to choose a gauge

Harmonic gauge (de Donder)

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\lambda{}_\lambda = 0 \quad \text{or} \quad \bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^\lambda{}_\lambda$$

$$\partial_\mu \bar{h}^{\mu\nu} = 0$$

Then

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda\right)$$

or

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{\kappa}{2} T_{\mu\nu}}$$

Solution for point mass $\quad T_{00} = M\delta^3(x)$, $T_{ij} = 0$

$$T_{\mu\nu} - \frac{1}{2}T^\lambda{}_\lambda = \frac{1}{2} M \delta^3(x) \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\Rightarrow \kappa h_{\mu\nu} = \begin{pmatrix} 2\phi_g & & & 0 \\ & 2\phi_g & & \\ & & 2\phi_g & \\ 0 & & & 2\phi_g \end{pmatrix} \quad \text{with} \quad \phi_g = -\frac{\kappa^2 M}{32\pi}\frac{1}{r} = -\frac{GM}{r}$$

Gauge invariance for the scalar field

$$\mathcal{L} = \frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right]$$

For small gauge trans

$$g'^{\mu\nu} = g^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu$$

$$\partial'_\mu = \partial_\mu - (\partial_\mu\xi^\lambda)\,\partial_\lambda$$

Then $\mathcal{L}' = \frac{1}{2}\left[(\eta^{\mu\nu} \qquad\qquad)(\eta_{\mu\alpha} - - \quad)\,\partial_\alpha\phi\,(\eta_{\nu\beta} \qquad)\,\partial_\beta\phi\right.$

$= \mathcal{L}$ ✓✓

Schroedinger Eq in gravitational field

Look at $(\Box + m^2)\phi = 0$

In harmonic coordinates

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\right) = g^{\mu\nu}\partial_\mu\partial_\nu + \underbrace{\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\right)}_{\substack{\partial_\mu\left[\left(1+\frac{\kappa}{2}h^\lambda_\lambda\right)\left(\eta^{\mu\nu}-\kappa h^{\mu\nu}\right)\right] \\ = \kappa\,\partial_\mu \bar{h}^{\mu\nu} = 0}}\partial_\nu$$

$$= g^{\mu\nu}\partial_\mu\partial_\nu$$

$$\Rightarrow \left[g^{\mu\nu}\partial_\mu\partial_\nu + m^2\right]\phi$$

where
$$g^{\mu\nu} = \begin{pmatrix} 1-2\phi_g & & & 0 \\ & -(1+2\phi_g) & & \\ & & -(1+2\phi_g) & \\ 0 & & & -(1+2\phi_g) \end{pmatrix}$$

Non relativistic reduction:

Let $\phi = e^{-imt}\psi(x,t)$

Then $\Big[\underbrace{g^{00}}_{m^2\phi^2}(-m^2 - 2im\partial_0 + \partial_0^2) - g^{ii}\nabla_i\nabla_i + m^2\Big]\psi = 0$

small $p^2\phi^2$

Drop $p^2\phi_g$ vs $m^2\phi_g$ $\Longrightarrow$ m^2 cancel $\times 1$

$\Rightarrow \left[m^2\, 2\phi_g - 2im\partial_0 - \nabla^2\right]\psi = 0$

Sch eq $\quad i\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\nabla^2}{2m} + m\phi_g\right]\psi(x,t)$

$-\frac{GMm}{r} \leftarrow V(r)$

Post Newtonian — Einstein Infeld Hoffman

$$
\begin{aligned}
H &= \left(\frac{\mathbf{p}^2}{2m_1}+\frac{\mathbf{p}^2}{2m_2}\right)-\left(\frac{\mathbf{p}^4}{8m_1^3}+\frac{\mathbf{p}^4}{8m_2^3}\right)\\
&\quad -\frac{Gm_1m_2}{r}\left[1+a\frac{\mathbf{p}^2}{m_1m_2}+b\frac{(\mathbf{p}\cdot\hat{\mathbf{r}})^2}{m_1m_2}+c\frac{G(m_1+m_2)}{r}\right]
\end{aligned}
$$

QFT

$a=\frac{1}{2}\left[1+3\frac{(m_1+m_2)^2}{m_1m_2}\right]$, $b=\frac{1}{2}$, $c=-\frac{1}{2}$ EIH coord.

Coord. trans.

$r \to r\left[1+\alpha\frac{G(m_1+m_2)}{r}\right]$ def of p changes also

Invariant under

$b \to \frac{1}{2}-\alpha\frac{(m_1+m_2)^2}{m_1m_2}$, $c=-\frac{1}{2}-\alpha$, $a\to\frac{1}{2}\left[1+(3+2\alpha)\left(\frac{(m_1+m_2)^2}{m_1m_2}\right)\right]$