Hawking Radiation As Tunneling

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We present a short and direct derivation of Hawking radiation as a tunneling process, based on particles in a dynamical geometry. The imaginary part of the action for the classically forbidden process is related to the Boltzmann factor for emission at the Hawking temperature. Because the derivation respects conservation laws, the exact spectrum is not precisely thermal. We compare and contrast the problem of spontaneous emission of charged particles from a charged conductor.

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I. Introduction.—Several derivations of Hawking radiation exist in the literature [1,2]. None of them, however, correspond very directly to one of the heuristic pictures most commonly proposed to visualize the source of the radiation, as tunneling. According to this picture, the radiation arises by a process similar to electron-positron pair creation in a constant electric field. The idea is that the energy of a particle changes sign as it crosses the horizon, so that a pair created just inside or just outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side.

Here we shall show that this schematic can be used to provide a short, direct semiclassical derivation of black hole radiance. In what follows, energy conservation plays a fundamental role: one must make a transition between states with the same total energy, and the mass of the residual hole must go down as it radiates. Indeed, it is precisely the possibility of lowering the black hole mass which ultimately drives the dynamics. This supports the idea that, in quantum gravity, black holes are properly regarded as highly excited states.

Broadly speaking, there are two standard approaches to Hawking radiation. In the first, one considers a collapse geometry. The response of external fields to this can be done explicitly or implicitly by abstracting appropriate boundary conditions. In the second, one treats the black hole immersed in a thermal bath. In this approach, one shows that (in general, metastable) equilibrium is possible. By detailed balance, this implies emission from the hole. In both of the standard calculations, the background geometry is considered fixed, and energy conservation is not enforced during the emission process.

Here we will consider a hole in empty Schwarzschild space, but with a dynamical geometry so as to enforce energy conservation. (Despite appearances, the geometry is not truly static, since there is no global Killing vector.) Because we are treating this aspect more realistically, we must—and do—find corrections to the standard results. These become quantitatively significant when the quantum of radiation carries a substantial fraction of the mass of the hole.

II. Tunneling.—To describe across-horizon phenomena, it is necessary to choose coordinates which, unlike Schwarzschild coordinates, are not singular at the horizon. A particularly suitable choice is obtained by introducing a time coordinate,

$$t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}, \qquad (1)$$

where t_s is Schwarzschild time. With this choice, the line element reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}\, dt\, dr + dr^2 + r^2 d\Omega^2. \qquad (2)$$

There is now no singularity at $r = 2M$, and the true character of the spacetime, as being stationary but not static, is manifest. These coordinates were first introduced by Painlevé [3] (who used them to criticize general relativity, for allowing singularities to come and go!). Their utility for studies of black hole quantum mechanics was emphasized more recently [4].

For our purposes, the crucial features of these coordinates are that they are stationary and nonsingular through the horizon. Thus it is possible to define an effective "vacuum" state of a quantum field by requiring that it annihilate modes which carry negative frequency with respect to t; such a state will look essentially empty (in any case, nonsingular) to a freely falling observer as he or she passes through the horizon. This vacuum differs strictly from the standard Unruh vacuum, defined by requiring positive frequency with respect to the Kruskal coordinate $U = -\sqrt{r - 2M} \exp(-\frac{t_s - r}{4M})$ [5]. The difference, however, shows up only in transients, and does not affect the late-time radiation.

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The radial null geodesics are given by

$$\dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}, \qquad (3)$$

with the upper (lower) sign in Eq. (3) corresponding to outgoing (ingoing) geodesics, under the implicit assumption that t increases towards the future. These equations are modified when the particle's self-gravitation is taken into account. Self-gravitating shells in Hamiltonian gravity were studied by Kraus and Wilczek [6]. They found that, when the black hole mass is held fixed and the total Arnowitt-Deser-Misner mass is allowed to vary, a shell of energy ω moves in the geodesics of a spacetime with M replaced by $M + \omega$. If instead we fix the total mass and allow the hole mass to fluctuate, then the shell of energy ω travels on the geodesics given by the line element

$$ds^2 = -\left(1 - \frac{2(M - \omega)}{r}\right)dt^2 + 2\sqrt{\frac{2(M - \omega)}{r}}\, dtdr + dr^2 + r^2 d\Omega^2, \qquad (4)$$

so we should use Eq. (3) with $M \to M - \omega$.

Since the typical wavelength of the radiation is of the order of the size of the black hole, one might doubt whether a point particle description is appropriate. However, when the outgoing wave is traced back towards the horizon, its wavelength, as measured by local fiducial observers, is ever increasingly blueshifted. Near the horizon, the radial wave number approaches infinity and the point particle, or WKB, approximation is justified.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} can be expressed as

$$\text{Im}S = \text{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} p_r\, dr = \text{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\int_0^{p_r} dp_r'\, dr. \qquad (5)$$

Remarkably, this can be evaluated without entering into the details of the solution, as follows. We multiply and divide the integrand by the two sides of Hamilton's equation $\dot{r} = +\frac{dH}{dp_r}|_r$, change the variable from momentum to energy, and switch the order of integration to obtain

$$\text{Im}S = \text{Im}\int_M^{M-\omega}\int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{\dot{r}}\, dH = \text{Im}\int_0^{+\omega}\int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}}(-d\omega'), \qquad (6)$$

where we have used the modified Eq. (3), and the minus sign appears because $H = M - \omega'$. But now the integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time (that is, into the lower half ω' plane). In this way we obtain

$$\text{Im}S = +4\pi\omega\left(M - \frac{\omega}{2}\right), \qquad (7)$$

provided $r_{\text{in}} > r_{\text{out}}$. To understand this ordering—which supplies the correct sign—we observe that when the integrals in Eq. (5) are not interchanged, and with the contour evaluated via the prescription $\omega \to \omega - i\epsilon$, we have

$$\text{Im}S = +\text{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2M'}{r}}}\, dr = \text{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} -\pi r\, dr. \qquad (8)$$

Hence $r_{\text{in}} = 2M$ and $r_{\text{out}} = 2(M - \omega)$. (Incidentally, by comparing the above equation with Eq. (5), we also find that $\text{Im}p_r = -\pi r$.) Although this radially inward motion appears at first sight to be classically allowed, it is nevertheless a classically forbidden trajectory because the apparent horizon is itself contracting. Thus, the limits on the integral indicate that, over the course of the classically forbidden trajectory, the outgoing particle starts from $r = 2M - \epsilon$, just inside the *initial* position of the horizon, and traverses the contracting horizon to materialize at $r = 2(M - \omega) + \epsilon$, just outside the *final* position of the horizon.

Alternatively, and along the same lines, Hawking radiation can also be regarded as pair creation *outside* the horizon, with the negative energy particle tunneling into the black hole. Since such a particle propagates backward in time, we have to reverse time in the equations of motion. From the line element, Eq. (2), we see that time reversal corresponds to $\sqrt{\frac{2M}{r}} \to -\sqrt{\frac{2M}{r}}$. Also, since the antiparticle sees a geometry of fixed black hole mass, the upshot of self-gravitation is to replace M by $M + \omega$, rather than $M - \omega$. Thus an ingoing negative energy particle has

$$\text{Im}S = \text{Im}\int_0^{-\omega}\int_{r_{out}}^{r_{in}} \frac{dr}{-1 + \sqrt{\frac{2(M+\omega')}{r}}}\, d\omega' = +4\pi\omega\left(M - \frac{\omega}{2}\right), \qquad (9)$$

where to obtain the last equation we have used Feynman's "hole theory" deformation of the contour: $\omega' \to \omega' + i\epsilon$.

Both channels—particle or antiparticle tunneling—contribute to the rate for the Hawking process so, in a more detailed calculation, one would have to add their amplitudes before squaring in order to obtain the semiclassical tunneling rate. Such considerations, however, only concern the prefactor. In either treatment, the exponential part of the semiclassical emission rate, in agreement with [7], is

$$\Gamma \sim e^{-2\,\text{Im}S} = e^{-8\pi\omega(M-\omega/2)} = e^{+\Delta S_{\text{B-H}}}, \qquad (10)$$

where we have expressed the result more naturally in terms of the change in the hole's Bekenstein-Hawking entropy, $S_{\text{B-H}}$. When the quadratic term is neglected, Eq. (10) reduces to a Boltzmann factor for a particle with energy ω at the inverse Hawking temperature $8\pi M$. The ω^2 correction arises from the physics of energy conservation, which (roughly speaking) self-consistently raises the effective temperature of the hole as it radiates. That the exact result must be correct can be seen on physical grounds by considering the limit in which the emitted particle carries away the entire mass and charge of the black hole (corresponding to the transmutation of the black hole into an outgoing shell). There can be only one such outgoing state. On the other hand, there are $\exp(S_{\text{B-H}})$ states in total. Statistical mechanics then asserts that the probability of finding a shell containing all of the mass of the black hole is proportional to $\exp(-S_{\text{B-H}})$, as above.

Following standard arguments, Eq. (10) with the quadratic term neglected implies the Planck spectral flux appropriate to an inverse temperature of $8\pi M$:

$$\rho(\omega) = \frac{d\omega}{2\pi} \frac{|T(\omega)|^2}{e^{+8\pi M\omega} - 1}, \tag{11}$$

where $|T(\omega)|^2$ is the frequency-dependent (graybody) transmission coefficient for the outgoing particle to reach future infinity without backscattering. It arises from a more complete treatment of the modes, whose semiclassical behavior near the turning point we have been discussing.

The preceding techniques can also be applied to emission from a charged black hole. However, when the outgoing radiation carries away the black hole's charge, the calculations are complicated by the fact that the trajectories are now also subject to electromagnetic forces. Here we restrict ourselves to uncharged radiation coming from a Reissner-Nordström black hole. The derivation then proceeds in a similar fashion to that above.

The charged counterpart to the Painlevé line element is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\, dt dr + dr^2 + r^2 d\Omega^2, \tag{12}$$

which is obtained from the standard Reissner-Nordström line element by the coordinate transformation,

$$t = t_r + 2\sqrt{2Mr - Q^2} + M \ln\left(\frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}}\right) + \frac{Q^2 - M^2}{\sqrt{M^2 - Q^2}} \operatorname{arctanh}\left(\frac{\sqrt{M^2 - Q^2}\sqrt{2Mr - Q^2}}{Mr}\right), \tag{13}$$

where t_r is the Reissner time coordinate. The line element now manifestly displays the stationary, nonstatic, and nonsingular nature of the spacetime.

The equation of motion for an outgoing massless particle is

$$\dot{r} \equiv \frac{dr}{dt} = +1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}, \tag{14}$$

with $M \to M - \omega$ when self-gravitation is included [8]. The imaginary part of the action for a positive energy outgoing particle is

$$\text{Im} S = \text{Im} \int_0^{+\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}} (-d\omega'), \tag{15}$$

which is again evaluated by deforming the contour in accordance with Feynman's $w' \to w' - i\epsilon$ prescription. The residue at the pole can be read off by substituting $u \equiv \sqrt{2(M - \omega')r - Q^2}$. This yields an emission rate of

$$\begin{aligned}\Gamma &\sim e^{-2\,\text{Im} S} \\ &= e^{-4\pi[2\omega(M-\omega/2) - (M-\omega)\sqrt{(M-\omega)^2 - Q^2} + M\sqrt{M^2 - Q^2}]} \\ &= e^{+\Delta S_{\text{B-H}}}.\end{aligned} \tag{16}$$

To first order in ω, Eq. (16) is consistent with Hawking's result of thermal emission at the Hawking temperature, T_H, for a charged black hole:

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}. \tag{17}$$

Again, energy conservation implies that the exact result has corrections of higher order in ω; these can all be collected to express the emission rate as the exponent of the change in entropy. Moreover, since the emission rate has to be real, the presence of the first square root in Eq. (16) ensures that radiation past extremality is not possible. Unlike in the traditional formulas, the third law of black hole thermodynamics is here manifestly enforced.

Note that only local physics has gone into our derivations. There was neither an appeal to Euclideanization nor any need to invoke an explicit collapse phase. The time asymmetry leading to outgoing radiation arose instead from use of the "normal" local contour deformation prescription in terms of the nonstatic coordinate t.

III. Relation to Electric Discharge.—The calculation presented above is formally self-contained, but some additional discussion is in order, to elucidate its physical meaning and to dispel a puzzle it poses.

When considered at the very broadest level, radiation of mass from a black hole resembles tunneling of electric charge off a charged conducting sphere. Upon a moment's reflection, however, the difference in the physics of the two situations appears so striking as to pose a puzzle. For while the electric force between like charges is repulsive, the gravitational force is always attractive. Related to this, the field energy of electric fields is positive, while (heuristically) the field energy of gravitational fields is negative. On this basis one might think that the electric process should proceed spontaneously, and need not require tunneling, while the gravitational process has no evident reason to proceed at all.

Consider a conducting sphere of radius R carrying charge Q. The electric field energy can be lowered by emitting a charge q so we expect this process to occur spontaneously. If we neglect backreaction of the charge

q on the conducting sphere, the force is repulsive at all distances, and there is no barrier to emission. In a more accurate treatment, however, we must take into account the induced nonuniformity of the charge on the sphere, which is easily done by using the method of images. The effective potential is

$$V(r) = q\left(\frac{Q - q}{r} - \frac{qR}{r^2 - R^2}\right), \qquad (18)$$

where we consider configurations of image charge which leave the potential on the sphere constant and the field at infinity fixed. In the formal limit $Q \gg q$ the first term dominates, and the potential decreases monotonically with r, indicating no barrier. However the second term increases monotonically with r, and always dominates for $r \rightarrow R$, producing a barrier.

In the gravitational problem, the situation is just the reverse. With backreaction neglected, there is nothing but barrier. Yet our calculation including backreaction indicates the possibility of redistributing mass energy of the gravitating sphere (black hole) into kinetic energy of emitted radiation.

Since the intrinsic energy of the gravitational field is negative, it is disadvantageous to reduce it, point by point. However, since in general the spacetime containing a black hole is not globally static, there exist freely propagating negative energy modes inside the horizon which cause the black hole to shrink. As a consequence, the black hole's radius decreases and *the external volume of space, over which one integrates the field, increases*. This, kinematically, is why the radiation process is allowed. Were the hole geometry to be regarded as fixed, there would be no possible source for the kinetic energy of the radiation, and a genuine tunneling interpretation of Hawking radiation would appear to be precluded.

IV. Conclusion.—We have derived Hawking radiation from the heuristically familiar perspective of tunneling. Our derivation is in consonance with intuitive notions of black hole radiance but, by taking into account global conservation laws, we are led to a modification of the emission spectrum away from thermality. The resulting corrected formula has physically reasonable limiting cases and, by virtue of nonthermality, suggests the possibility of information-carrying correlations in the radiation.

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