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The one-loop effective action and trace anomaly in four dimensions

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Abstract

The one-loop effective action for a generic set of quantum fields is calculated as a nonlocal expansion in powers of the curvatures (field strengths). This expansion is obtained to third order in the curvature. It is stressed that the covariant vertices are finite. The trace anomaly in four dimensions is obtained directly by varying the effective action. The nonlocal terms in the action, producing the anomaly, contain nontrivial functions of three operator arguments. The trace anomaly is derived also by making the conformal transformation in the heat kernel.

1. Introduction

The trace anomaly in the vacuum energy–momentum tensor of conformal invariant fields discovered long ago [1–4] has important applications [5–8] and presently arouses a new interest [9–15]. The recent studies are motivated by the wish to extend two-dimensional results [11] and by the fact that the anomaly in four dimensions may be associated with important physical effects [16,7,17,10,12].

The most interesting feature of the trace anomaly is that it contains information about the nonlocal structure of the effective action. It is the nonlocal effective

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action that gives rise to the physical effects [7,5,18,19,8,12,20–22]. Unlike in two dimensions [16,5,7,18], in higher dimensions the trace anomaly does not control the effective action completely. Two actions producing one and the same anomaly differ generally by an arbitrary conformal invariant functional. Nevertheless, the trace anomaly may help to build a model effective action that would contain the essential nonlocal effects and could be used for studying the field's expectation values in four and higher dimensions.

Although there have been studies aimed at building nonlocal geometric invariants associated with the anomaly [23,9,11,13], the latter has never been derived from the effective action directly. The purpose of the present work is to fill this gap. As will be seen, the nonlocal structures that really appear in the effective action and give rise to the anomaly are more complicated than the ones used in the above mentioned constructions.

The one-loop effective action for a generic set of quantum fields has recently been calculated to third order in the curvature [24]. The method for this calculation, the covariant perturbation theory, was developed in [25,10,26]. In this method, the effective action is obtained as a nonlocal expansion in powers of a universal set of field strengths (curvatures) characterizing a generic field model. To reproduce the trace anomaly in four dimensions, one needs this expansion to third order. Here we shall use the results of the report [24] to display the mechanism by which the local trace anomaly emerges from the nonlocal effective action.

The plan of the paper is as follows. In Section 2, we present the form of the nonlocal expansion for the effective action to third order in the curvature, and comment on its derivation by covariant perturbation theory. We discuss also the mechanism by which the finite covariant vertices are obtained. Section 3 contains a direct derivation of the trace anomaly from the nonlocal effective action. We show, in particular, that quadratic (in the curvature) terms of the action do not generate a local anomaly. Their contribution to the anomaly, even to the lowest nontrivial order, contains a certain nonlocal form factor which is cancelled only after adding the contribution of the covariant vertices (cf. [13]). In Section 4, the trace anomaly is derived by making the conformal transformation of the heat kernel. In this derivation, we use the results of calculation of the heat kernel by covariant perturbation theory [10,24,15]. We show that the mechanism of the cancellation of nonlocal terms leading to the local anomaly is the same as in two dimensions [15]. Namely, after the conformal transformation the heat kernel becomes a total derivative, and the anomaly is determined only by the early-time asymptotic behaviour.

2. The effective action in covariant perturbation theory

Here we consider the effective action in noncompact asymptotically flat and empty spacetime with the euclidean (positive-signature) metric, which is sufficient for finding the expectation-value equations of lorentzian field theory with the *in*-vacuum quantum state [7,10]. For a generic quantum field model it can be obtained by calculating loops with the inverse propagator – the second-order operator

$$H = g^{\mu\nu}\nabla_\mu\nabla_\nu + \left(\hat{P} - \tfrac{1}{6}R\hat{1}\right), \tag{2.1}$$

acting on an arbitrary set of fields $\varphi^A(x)$. Here A stands for any set of discrete indices, and the hat indicates that the quantity is a matrix in the corresponding vector space of field components: $\hat{1} = \delta^A{}_B$, $\hat{P} = P^A{}_B$, etc.. Below, the matrix trace operation will be denoted by tr: tr $\hat{1} = \delta^A{}_A$, tr $\hat{P} = P^A{}_A$, etc.. In (1.2), $g_{\mu\nu}$ is a positive-definite metric characterized by its Riemann curvature $R^{\alpha\beta\mu\nu}$, for which we use the sign convention $R^\mu_{\alpha\nu\beta} = \partial_\nu\Gamma^\mu_{\alpha\beta} - \dots$, $R_{\alpha\beta} = R^\mu_{\alpha\mu\beta}$, $R = g^{\alpha\beta}R_{\alpha\beta}$, ∇_μ is a covariant derivative (with respect to an arbitrary connection) characterized by its commutator curvature

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\varphi^A = \mathcal{R}^A{}_{B\mu\nu}\varphi^B, \quad \mathcal{R}^A{}_{B\mu\nu} \equiv \hat{\mathcal{R}}_{\mu\nu}, \tag{2.2}$$

and $\hat{P}$ is an arbitrary matrix. The redefinition of the potential in (2.1) by inclusion of the term in the Ricci scalar R is a matter of convenience.

There are three independent inputs in the operator (2.1): $g^{\mu\nu}$, ∇_μ and $\hat{P}$ – the metric contracting the second derivatives, the connection which defines the covariant derivative, and the potential matrix. They may be regarded as background fields to which correspond their field strengths or curvatures. There is the Riemann curvature associated with $g_{\mu\nu}$, the commutator curvature (2.2) associated with ∇_μ, and the potential $\hat{P}$ which is its own "curvature". An important feature of the asymptotically flat spacetime is that its purely gravitational strength boils down to the Ricci tensor, because the Riemann tensor can always be eliminated via the differentiated Bianchi identity [10,27] by iteratively solving it for $R^{\alpha\beta\mu\nu}$ in terms of $R^{\mu\nu}$. This iterational solution is uniquely determined by the Green function $1/\Box$ with zero boundary conditions at infinity and starts with

$$\begin{aligned} R^{\alpha\beta\mu\nu} = \frac{1}{2}\frac{1}{\Box}\big(&\nabla^\mu\nabla^\alpha R^{\nu\beta} + \nabla^\alpha\nabla^\mu R^{\nu\beta} - \nabla^\nu\nabla^\alpha R^{\mu\beta} - \nabla^\alpha\nabla^\nu R^{\mu\beta} - \nabla^\mu\nabla^\beta R^{\nu\alpha} \\ &- \nabla^\beta\nabla^\mu R^{\nu\alpha} + \nabla^\nu\nabla^\beta R^{\mu\alpha} + \nabla^\beta\nabla^\nu R^{\mu\alpha}\big) + \mathrm{O}\big(R^2_{..}\big). \end{aligned} \tag{2.3}$$

Thus, the full set of field strengths characterizing the operator (2.1), for which we shall use the collective notation $\mathfrak{R}$, includes

$$\mathfrak{R} = \left(R_{\mu\nu}, \hat{\mathcal{R}}_{\mu\nu}, \hat{P}\right). \tag{2.4}$$

At one-loop order the effective action is given by the expression

$$-W = -\tfrac{1}{2}\mathrm{Tr}\ln H + \int d^4x\, \delta^4(x,x)(\dots), \tag{2.5}$$

where Tr as distinct from tr denotes the functional trace, and the term with ellipses $(\dots)$ stands for the contribution of the local functional measure, proportional to the delta-function at coincident points. As shown in [28], this contribution always cancels the volume divergences of the loop which otherwise would appear in (2.5) in the form of a divergent cosmological term. *For a massless operator* (2.1), the result of this cancellation is a subtraction of the term of zeroth order in the curvature. The masslessness of the operator (2.1) means that, like the Riemann and commutator curvatures, the potential $\hat{P}$ falls off at infinity of the manifold. For the precise conditions of this falloff see [10].

The effective action (2.5) is an invariant functional of background fields. In the four-dimensional euclidean asymptotically flat spacetime, it is analytic in the curvatures (2.4) [15]. It is, therefore, expandable in the basis of nonlocal invariants of Nth order in $\Re$ built in [27] for $N = 1$, 2 and 3. In covariant perturbation theory, the expansion goes in powers of these curvatures, but its coefficients (the nonlocal form factors) are also field dependent. The meaning of such an expansion is that the effective action is obtained with accuracy $\mathrm{O}(\Re^N)$, i.e. *up to* Nth power in $\Re$. Each term of the expansion contains N curvatures $\Re$ *explicitly* and is defined up to $\mathrm{O}(\Re^{N+1})$.

The result for the one-loop effective action (2.5) to third order in the curvature is of the form [24]

$$-W = \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}\, \mathrm{tb}\Bigg(\sum_{i=1}^{5}\gamma_i(-\Box_2)\Re_1\Re_2(i)$$

$$+\sum_{i=1}^{29}\Gamma_i(-\Box_1,-\Box_2,-\Box_3)\Re_1\Re_2\Re_3(i)+\mathrm{O}(\Re^4)\Bigg). \tag{2.6}$$

Here terms of zeroth and first order in the curvature are omitted. [4] The quadratic $\Re_1\Re_2(i)$, $i = 1$ to 5, and cubic $\Re_1\Re_2\Re_3(i)$, $i = 1$ to 29, curvature invariants here, as well as the conventions concerning the action of covariant D'Alambertian arguments ($\Box_1$, $\Box_2$, $\Box_3$) on the curvatures ($\Re_1$, $\Re_2$, $\Re_3$) labelled by the corresponding numbers, are presented and discussed in detail in [10,27]. Terms of second order in the curvature are given by five quadratic structures

$$\begin{aligned}
\Re_1\Re_2(1) &= R_{1\mu\nu}R_2^{\mu\nu}\hat{1},\\
\Re_1\Re_2(2) &= R_1R_2\hat{1},\\
\Re_1\Re_2(3) &= \hat{P}_1R_2,\\
\Re_1\Re_2(4) &= \hat{P}_1\hat{P}_2,\\
\Re_1\Re_2(5) &= \hat{\mathscr{R}}_{1\mu\nu}\hat{\mathscr{R}}_2^{\mu\nu},
\end{aligned} \tag{2.7}$$

[4] Since these terms are local and at most quadratic in derivatives, they must be removed by renormalization. The zeroth-order term violates the boundary condition of asymptotic flatness but, in the present case of massless quantum fields, it is cancelled by the contribution of the functional measure (see [28]).

and terms of third order by 29 cubic structures $\Re_1\Re_2\Re_3(i)$, $i = 1, \ldots, 29$. Eleven of them contain no derivatives

$$
\begin{aligned}
\Re_1\Re_2\Re_3(1) &= \hat{P}_1\hat{P}_2\hat{P}_3,\\
\Re_1\Re_2\Re_3(2) &= \hat{\mathcal{R}}^{\mu}_{1\ \alpha}\hat{\mathcal{R}}^{\alpha}_{2\beta}\hat{\mathcal{R}}^{\beta}_{3\ \mu},\\
\Re_1\Re_2\Re_3(3) &= \hat{\mathcal{R}}^{\mu\nu}_{1}\hat{\mathcal{R}}_{2\mu\nu}\hat{P}_3,\\
\Re_1\Re_2\Re_3(4) &= R_1R_2\hat{P}_3,\\
\Re_1\Re_2\Re_3(5) &= R^{\mu\nu}_1R_{2\mu\nu}\hat{P}_3,\\
\Re_1\Re_2\Re_3(6) &= \hat{P}_1\hat{P}_2R_3,\\
\Re_1\Re_2\Re_3(7) &= R_1\hat{\mathcal{R}}^{\mu\nu}_2\hat{\mathcal{R}}_{3\mu\nu},\\
\Re_1\Re_2\Re_3(8) &= R^{\alpha\beta}_1\hat{\mathcal{R}}_{2\alpha}^{\ \ \mu}\hat{\mathcal{R}}_{3\beta\mu},\\
\Re_1\Re_2\Re_3(9) &= R_1R_2R_3\hat{1},\\
\Re_1\Re_2\Re_3(10) &= R^{\mu}_{1\alpha}R^{\alpha}_{2\beta}R^{\beta}_{3\mu}\hat{1},\\
\Re_1\Re_2\Re_3(11) &= R^{\mu\nu}_1R_{2\mu\nu}R_3\hat{1},
\end{aligned}
\tag{2.8}
$$

Fourteen structures contain two derivatives,

$$
\begin{aligned}
\Re_1\Re_2\Re_3(12) &= \hat{\mathcal{R}}^{\alpha\beta}_1\nabla^{\mu}\hat{\mathcal{R}}_{2\mu\alpha}\,\nabla^{\nu}\hat{\mathcal{R}}_{3\nu\beta},\\
\Re_1\Re_2\Re_3(13) &= \hat{\mathcal{R}}^{\mu\nu}_1\nabla_{\mu}\hat{P}_2\,\nabla_{\nu}\hat{P}_3,\\
\Re_1\Re_2\Re_3(14) &= \nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}_1\,\nabla^{\nu}\hat{\mathcal{R}}_{2\nu\alpha}\,\hat{P}_3,\\
\Re_1\Re_2\Re_3(15) &= R^{\mu\nu}_1\nabla_{\mu}R_2\,\nabla_{\nu}\hat{P}_3,\\
\Re_1\Re_2\Re_3(16) &= \nabla^{\mu}R^{\nu\alpha}_1\,\nabla_{\nu}R_{2\mu\alpha}\,\hat{P}_3,\\
\Re_1\Re_2\Re_3(17) &= R^{\mu\nu}_1\nabla_{\mu}\nabla_{\nu}\hat{P}_2\,\hat{P}_3,\\
\Re_1\Re_2\Re_3(18) &= R_{1\alpha\beta}\nabla_{\mu}\hat{\mathcal{R}}^{\mu\alpha}_2\,\nabla_{\nu}\hat{\mathcal{R}}^{\nu\beta}_3,\\
\Re_1\Re_2\Re_3(19) &= R^{\alpha\beta}_1\nabla_{\alpha}\hat{\mathcal{R}}^{\mu\nu}_2\,\nabla_{\beta}\hat{\mathcal{R}}_{3\mu\nu},\\
\Re_1\Re_2\Re_3(20) &= R_1\nabla_{\alpha}\hat{\mathcal{R}}^{\alpha\mu}_2\,\nabla^{\beta}\hat{\mathcal{R}}_{3\beta\mu},\\
\Re_1\Re_2\Re_3(21) &= R^{\mu\nu}_1\nabla_{\mu}\nabla_{\lambda}\hat{\mathcal{R}}^{\lambda\alpha}_2\,\hat{\mathcal{R}}_{3\alpha\nu},\\
\Re_1\Re_2\Re_3(22) &= R^{\alpha\beta}_1\nabla_{\alpha}R_2\,\nabla_{\beta}R_3\,\hat{1},\\
\Re_1\Re_2\Re_3(23) &= \nabla^{\mu}R^{\nu\alpha}_1\,\nabla_{\nu}R_{2\mu\alpha}\,R_3\hat{1},
\end{aligned}
\tag{2.9}
$$

$$\Re_1\Re_2\Re_3(24) = R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\alpha\beta}\hat{1},$$

$$\Re_1\Re_2\Re_3(25) = R_1^{\mu\nu}\nabla_\alpha R_{2\beta\mu}\nabla^\beta R_{3\nu}^\alpha\hat{1},$$

three structures contain four derivatives

$$\begin{aligned}
\Re_1\Re_2\Re_3(26) &= \nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}\hat{P}_3,\\
\Re_1\Re_2\Re_3(27) &= \nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta} R_3\hat{1},\\
\Re_1\Re_2\Re_3(28) &= \nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R_{2\lambda}^{\beta}\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1},
\end{aligned} \tag{2.10}$$

and one structure contains six derivatives

$$\Re_1\Re_2\Re_3(29) = \nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma}\hat{1}. \tag{2.11}$$

Here we present only those elements of the full basis of third-order invariants explicitly built in [27], whose coefficients are nonvanishing in the one-loop effective action. It turns out that these basis elements are also not completely independent, and there exists a nonlocal identity [24,27]

$$\begin{aligned}
\int d^4x\, g^{1/2}\,\mathrm{tr}\mathscr{F}^{\mathrm{sym}}(\Box_1,\Box_2,\Box_3)\Big[&-\tfrac{1}{48}\big(\Box_1^{\,2}+\Box_2^{\,2}+\Box_3^{\,2}\big)\Re_1\Re_2\Re_3(9)\\
&-\tfrac{1}{12}\big(\Box_1^{\,2}+\Box_2^{\,2}+\Box_3^{\,2}-2\Box_1\Box_2-2\Box_2\Box_3-2\Box_1\Box_3\big)\Re_1\Re_2\Re_3(10)\\
&-\tfrac{1}{8}\Box_3(\Box_1+\Box_2-\Box_3)\Re_1\Re_2\Re_3(11)\\
&+\tfrac{1}{8}(3\Box_1+\Box_2+\Box_3)\Re_1\Re_2\Re_3(22)-\tfrac{1}{2}\Box_3\Re_1\Re_2\Re_3(23)\\
&-\tfrac{1}{2}\Box_1\Re_1\Re_2\Re_3(24)-\tfrac{1}{2}(\Box_2+\Box_3-\Box_1)\Re_1\Re_2\Re_3(25)\\
&+\tfrac{1}{2}\Re_1\Re_2\Re_3(27)+\Re_1\Re_2\Re_3(28)\Big]+\mathrm{O}(\Re^4)=0,
\end{aligned} \tag{2.12}$$

valid in four-dimensional asymptotically flat spacetime with an arbitrary coefficient function $\mathscr{F}^{\mathrm{sym}}(\Box_1,\Box_2,\Box_3)$ *completely symmetric* in its operator box arguments. This identity will be used to exclude the completely symmetric (under the permutation of labels 1,2,3) part of the structure $\Re_1\Re_2\Re_3$ (28).

The operator coefficients in (2.6) – the nonlocal form factors of the effective action – are obtained in covariant perturbation theory [10] from the analogous nonlocal form factors in the trace of the heat kernel $\mathrm{Tr}\,K(s) = \mathrm{Tr}\exp(sH)$ by integrating over the proper time according to the equation

$$\tfrac{1}{2}\mathrm{Tr}\ln H = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}\,K(s). \tag{2.13}$$

The trace of the heat kernel calculated to third order in the curvature in [24]

$$\begin{aligned}\mathrm{Tr}\,K(s) = \frac{1}{(4\pi s)^{\omega}}\int dx\, g^{1/2}\,\mathrm{tr}\Bigg(\hat{1} + s\hat{P} + s^2\sum_{i=1}^{5} f_i(-s\,\Box_2)\Re_1\Re_2(i)\\ + s^3\sum_{i=1}^{11} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3)\Re_1\Re_2\Re_3(i)\\ + s^4\sum_{i=12}^{25} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3)\Re_1\Re_2\Re_3(i)\\ + s^5\sum_{i=26}^{28} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3)\Re_1\Re_2\Re_3(i)\\ + s^6 F_{29}(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3)\Re_1\Re_2\Re_3(29) + \mathrm{O}(\Re^4)\Bigg)\end{aligned} \tag{2.14}$$

thus generates the second-order and the third-order form factors in Eq. (2.6),

$$\gamma_i(-\Box_2) = (4\pi)^2\int_0^\infty \frac{ds}{s}\frac{s^2}{(4\pi s)^{\omega}} f_i(-s\,\Box_2), \tag{2.15}$$

$$\Gamma_i(-\Box_1, -\Box_2, -\Box_3) = (4\pi)^2\int_0^\infty \frac{ds}{s}\frac{s^{p_i}}{(4\pi s)^{\omega}} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3), \tag{2.16}$$

with $p_i = 3$ for $i = 1$ to 11, $p_i = 4$ for $i = 12$ to 25, $p_i = 5$ for $i = 26$ to 28 and $p_i = 6$ for $i = 29$. Here 2ω is the spacetime dimension which plays the role of the parameter in the dimensional regularization of ultraviolet divergences at $\omega \to 2$. Simple expressions for $f_i(-s\,\Box_2)$ obtained in [10] give rise to second-order form factors

$$\begin{aligned}\gamma_1(-\Box) &= \frac{1}{60}\left[-\ln\left(-\frac{\Box}{\mu^2}\right) + \frac{16}{15}\right],\\ \gamma_2(-\Box) &= \frac{1}{180}\left[\ln\left(-\frac{\Box}{\mu^2}\right) - \frac{37}{30}\right],\\ \gamma_3(-\Box) &= -\tfrac{1}{18},\\ \gamma_4(-\Box) &= -\frac{1}{2}\ln\left(-\frac{\Box}{\mu^2}\right),\\ \gamma_5(-\Box) &= \frac{1}{12}\left[-\ln\left(-\frac{\Box}{\mu^2}\right) + \frac{2}{3}\right],\end{aligned} \tag{2.17}$$

where the parameter $\mu^2 > 0$ accounts for the ultraviolet arbitrariness in all the form factors except $\gamma_3(-\Box)$ which is local and independent of μ^2 (see [10]). The arbitrariness in μ^2 results from subtracting the logarithmic divergences of the effective action, accumulating the pole parts of the divergent integrals (2.15). They are given by the integrated DeWitt coefficient $a_2(x,x)$ at coincident points [29,30],

$$-W^{\text{div}}=\left(\frac{1}{2-\omega}+\ln(4\pi)-C+2\right)\int \mathrm{d}^4x g^{1/2}\,\mathrm{tr}\,\hat{a}_2(x,x),\quad \omega\to 2, \tag{2.18}$$

$$\hat{a}_2(x,x)=\tfrac{1}{180}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-R_{\mu\nu}R^{\mu\nu}\right)\hat{1}+\tfrac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}+\tfrac{1}{2}\hat{P}^2+\tfrac{1}{6}\Box\hat{P}$$
$$+\tfrac{1}{180}\Box R\hat{1}, \tag{2.19}$$

which absorbs all the divergences of (2.15) on account of the Gauss–Bonnet identity

$$\int \mathrm{d}^4x g^{1/2}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2\right)=0. \tag{2.20}$$

In contrast to (2.17), the third-order form factors contain no arbitrary parameters and are finite. This goes as follows. The integrals (2.16) are generally divergent, but the divergences cancel in the sum

$$\sum_i \Gamma_i(-\Box_1,-\Box_2,-\Box_3)\Re_1\Re_2\Re_3(i). \tag{2.21}$$

The mechanism of cancellation is the identity (2.12). If, in this identity, one puts

$$\mathcal{F}^{\text{sym}}(\Box_1,\Box_2,\Box_3)$$
$$=-\frac{2}{45}\frac{1}{\Box_1\Box_2\Box_3}\left(\frac{1}{2-\omega}+\ln(4\pi)-C-\tfrac{1}{3}\sum_{n=1}^{3}\ln(-\Box_n)\right), \tag{2.22}$$

then its left-hand side will be precisely the divergent term of (2.21). The reason why these divergences appear at all is the presence of the Riemann tensor in the DeWitt coefficient (2.19) which governs the ultraviolet divergences in four dimensions. Since covariant perturbation theory expands the Riemann tensor in an infinite series in powers of the Ricci tensor (see [10,24,27] and Eq. (2.3) above), the divergent term with the Riemann tensor brings divergent contributions to the third and all higher orders in the Ricci curvature. The problem vanishes, however, if one takes into account the Gauss–Bonnet identity (2.20) which, in four dimensions, eliminates the Riemann tensor from (the integrated) $a_2(x,x)$. Automatically eliminated then are also all divergent contributions of higher orders in the curvature. At each order, there exists a nonlocal constraint which ensures this elimination. The hierarchy of these constraints is generated by the expansion of the Gauss–Bonnet invariant (see Eq. (6.36) of [27] and the detailed derivation in the report [24]).

It is worth emphasizing the difference between covariant perturbation theory [10,24] and the usual flat-space perturbation theory. In the latter, noncovariant vertices *of all orders* are divergent. The divergences, therefore, proliferate in an apparently uncontrollable way and the fact that they cancel in some effect (see, e.g., [31]) looks like a miracle. In essence, no miracle happens. The physical effects are covariant and are determined by covariant vertices, i.e. form factors of W

starting with $N=3$, which are finite. A similar situation takes place with the infrared renormalization in quantum electrodynamics [19]. Covariant renormalization theory for gauge fields was pioneered by DeWitt [29]. For its extension beyond the one-loop level see [32].

After the use of the identity (2.12), the third-order form factors $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$, $i = 1$ to 29, when calculated by Eq. (2.16) from the form factors of the heat kernel [24], are all linearly expressed through the basic third-order form factor

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_{\alpha \geqslant 0} \mathrm{d}^3\alpha \frac{\delta(1-\alpha_1-\alpha_2-\alpha_3)}{-\alpha_1\alpha_2\Box_3 - \alpha_1\alpha_3\Box_2 - \alpha_2\alpha_3\Box_1} \tag{2.23}$$

and the second-order form factors

$$(\mathrm{a}) \quad \ln\frac{\Box_n}{\Box_m}, \qquad (b) \quad \frac{\ln(\Box_n/\Box_m)}{\Box_n - \Box_m}, \quad n,m = 1,2,3. \tag{2.24}$$

The coefficients of these expressions are rational functions of the following general form:

$$\frac{P(\Box)}{D^6\Box_1\Box_2\Box_3}, \tag{2.25}$$

where $P(\Box)$ is a polynomial and

$$D = \Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1\Box_2 - 2\Box_1\Box_3 - 2\Box_2\Box_3. \tag{2.26}$$

The basic third-order form factor $\Gamma(-\Box_1, -\Box_2, -\Box_3)$ is a solution of the differential equation

$$\frac{\partial}{\partial\Box_1}\Gamma = \frac{\Box_2 + \Box_3 - \Box_1}{D}\Gamma + \frac{1}{D}\ln\frac{\Box_2}{\Box_1} + \frac{1}{D}\ln\frac{\Box_3}{\Box_1} + \frac{\Box_2 - \Box_3}{D\Box_1}\ln\frac{\Box_2}{\Box_3} \tag{2.27}$$

and two other equations, with $\partial/\partial\Box_2$ and $\partial/\partial\Box_3$, obtained from (2.27) by cyclic symmetry. It also has the original α-representation (2.23), the Laplace representation [24]

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty \mathrm{d}^3u \frac{\exp(u_1\Box_1 + u_2\Box_2 + u_3\Box_3)}{u_1u_2 + u_2u_3 + u_1u_3}, \tag{2.28}$$

the spectral representation [26,24]

$$\Gamma(-\Box_1, -\Box_2, -\Box_3) = \int_0^\infty \frac{\mathrm{d}m_1^2\,\mathrm{d}m_2^2\,\mathrm{d}m_3^2\,\rho(m_1,\ m_2,\ m_3)}{(m_1^2 - \Box_1)(m_2^2 - \Box_2)(m_3^2 - \Box_3)} \tag{2.29}$$

with the discontinuous spectral weight $\rho(m_1,\ m_2,\ m_3)$ as a function of masses

$$m_1 = \sqrt{m_1^2},\ m_2 = \sqrt{m_2^2},\ m_3 = \sqrt{m_3^2},$$

$$\rho(m_1,\ m_2,\ m_3) = \frac{\theta(m_1+m_2-m_3)\theta(m_1+m_3-m_2)\theta(m_3+m_2-m_1)}{\pi\left[(m_1+m_2+m_3)(m_1+m_2-m_3)(m_1+m_3-m_2)(m_3+m_2-m_1)\right]^{1/2}}, \tag{2.30}$$

and the generalized spectral representation [24], based on the equation for Bessel functions $\int_0^\infty \mathrm{d}y^2 J_0(ym_1)J_0(ym_2)J_0(ym_3) = 4\rho(m_1, m_2, m_3)$,

$$\Gamma(-\Box_1, -\Box_2, -\Box_3)$$
$$= 2\int_0^\infty \mathrm{d}y^2 \mathscr{K}_0\big(y\sqrt{-\Box_1}\big)\mathscr{K}_0\big(y\sqrt{-\Box_2}\big)\mathscr{K}_0\big(y\sqrt{-\Box_3}\big), \tag{2.31}$$

$$\mathscr{K}_0\big(y\sqrt{-\Box}\big) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}m^2 J_0(ym)}{m^2 - \Box}. \tag{2.32}$$

The spectral representations are important for imposing the boundary conditions in the expectation-value equations of lorentzian spacetime [7,10].

Explicit expressions for all 29 form factors $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$ in terms of $\Gamma(-\Box_1, -\Box_2, -\Box_3)$ are presented in the report [24]. These expressions are cumbersome and are manageable only in the format of the computer algebra program *Mathematica* [5]. Therefore, they will not be presented here. The α-representation, Laplace representation and generalized spectral representation are also obtained in this report for all form factors $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$, as well as the asymptotic behaviours of these form factors at large and small values of their arguments. The small-$\Box$ behaviours [22] determine the vacuum radiation effect at null infinity [21].

3. Derivation of the trace anomaly

If the operator H in (2.1) corresponds to a conformal invariant quantum field in four dimensions, the trace anomaly should follow from (2.6) simply by varying W and taking the trace. Obtaining the correct trace anomaly is also a powerful check on the result for W. To have as many curvature structures as possible involved in the check, we choose the following quantum field model:

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}x\, g^{1/2}\left(\nabla_\mu\varphi^{\mathrm{T}}\nabla^\mu\varphi + \frac{R}{6}\varphi^{\mathrm{T}}\varphi + \frac{\lambda^2}{4!}\left(\varphi^{\mathrm{T}}\varphi\right)^2\right), \tag{3.1}$$

$$\nabla_\mu\varphi = \partial_\mu\varphi + A_\mu\hat{G}\varphi, \quad \nabla_\mu\varphi^{\mathrm{T}} = \partial_\mu\varphi^{\mathrm{T}} + A_\mu\varphi^{\mathrm{T}}\hat{G}^{\mathrm{T}}, \tag{3.2}$$

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad \hat{G} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{3.3}$$

where (3.1) is the euclidean action of the complex scalar quantum field $\varphi = \varphi_1 + i\varphi_2$, rewritten in terms of the real components, and T denotes the matrix transposition. The electromagnetic and gravitational fields in (3.1) are classical.

[5] These files are available from the authors.

The action (3.1) is invariant under the local conformal transformations

$$\delta_\sigma g^{\mu\nu}(x) = \sigma(x) g^{\mu\nu}(x), \quad \delta_\sigma \varphi(x) = \tfrac{1}{2}\sigma(x)\varphi(x), \quad \delta_\sigma A_\mu(x) = 0, \tag{3.4}$$

with the parameter $\sigma(x)$. The hessian of the action has the form (2.1) (times a local matrix) in which the potential and the commutator curvature are

$$\hat{P} = -\frac{2\lambda^2}{4!}\begin{pmatrix} 3\varphi_1^2 + \varphi_2^2 & 2\varphi_1\varphi_2 \\ 2\varphi_1\varphi_2 & 3\varphi_2^2 + \varphi_1^2 \end{pmatrix}, \tag{3.5}$$

$$\hat{\mathscr{R}}_{\mu\nu} = \hat{G}(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{3.6}$$

with $\hat{G}$ in (3.3).

From (3.4)–(3.6) we find the conformal transformation laws for the curvatures and $\Box$-operators:

$$\begin{aligned}
&\delta_\sigma \hat{P} = \sigma \hat{P}, \quad \delta_\sigma \hat{\mathscr{R}}_{\mu\nu} = 0, \\
&\delta_\sigma R_{\mu\nu} = \left(\nabla_\mu \nabla_\nu + \tfrac{1}{2} g_{\mu\nu} \Box\right)\sigma, \quad \delta_\sigma R = (3\Box + R)\sigma, \\
&(\delta_\sigma \Box)\hat{P} = \sigma \Box \hat{P} - \nabla_\alpha \sigma \nabla^\alpha \hat{P}, \\
&(\delta_\sigma \Box)\hat{\mathscr{R}}_{\mu\nu} = \sigma \Box \hat{\mathscr{R}}_{\mu\nu} + \hat{\mathscr{R}}_{\mu\nu} \Box \sigma + \nabla_\mu \sigma \nabla^\alpha \hat{\mathscr{R}}_{\alpha\nu} - \nabla_\nu \sigma \nabla^\alpha \hat{\mathscr{R}}_{\alpha\mu}, \\
&(\delta_\sigma \Box) R_{\mu\nu} = \sigma \Box R_{\mu\nu} + R_{\mu\nu} \Box \sigma + \nabla_\alpha \sigma \nabla^\alpha R_{\mu\nu} + \nabla_{(\mu} \sigma \nabla_{\nu)} R - 2\nabla^\alpha \sigma \nabla_{(\mu} R_{\nu)\alpha}, \\
&(\delta_\sigma \Box) R = \sigma \Box R - \nabla_\alpha \sigma \nabla^\alpha R.
\end{aligned} \tag{3.7}$$

Having got these laws, one may already forget the particular content of the model, and merely consider the transformation (3.7) in the effective action. For the dimensionally regularized [6] one-loop effective action (2.6), the result should be exactly

$$-\delta_\sigma W = -\frac{1}{2(4\pi)^2}\int dx\, g^{1/2} \sigma(x)\, \mathrm{tr}\, \hat{a}_2(x,x), \tag{3.8}$$

where $\hat{a}_2(x,x)$ is the second DeWitt coefficient at coincident points (2.19) [7]. Expression (3.8) is the general form of the conformal anomaly in four dimensions [1–4]. For the model above,

$$\delta_\sigma = \int dx \left(\sigma(x) g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + \frac{1}{2}\sigma(x)\varphi \frac{\delta}{\delta\varphi} \right) \tag{3.9}$$

[6] The dimensional regularization was used for the derivation of the quadratic terms in W (see Section 2 and [10]). In fact, important is only the belonging of the regularization to one of the two alternative classes discussed in [4].

[7] Since the function $\sigma(x)$ is arbitrary in any compact domain, the anomaly (3.8) provides a check of $\mathrm{tr}\hat{a}_2(x,x)$ itself whereas in Section 2 we dealt with the integral $\int dx g^{1/2}\, \mathrm{tr}\, \hat{a}_2(x,x)$, discarding possible total derivative terms of $\mathrm{tr}\, \hat{a}_2(x,x)$. See also [33] for the restoration of such terms from integral quantities by the variational method.

and

$$g^{-1/2}\left(g^{\mu\nu}\frac{\delta W}{\delta g^{\mu\nu}}+\frac{1}{2}\varphi\frac{\delta W}{\delta\varphi}\right)=\frac{1}{2(4\pi)^2}\operatorname{tr}\hat{a}_2(x,x). \tag{3.10}$$

In the present technique, Eq. (3.8) with $\hat{a}_2(x,x)$ given by (2.19) can be obtained only with a given accuracy $O(\Re^n)$ and with the Riemann tensor expressed through the Ricci tensor. To lowest order, one may use the expression for $R^2_{\alpha\beta\mu\nu}$ given in Appendix A of [10]. After elimination of the Riemann tensor from (2.19), Eq. (3.8) takes the form

$$\begin{aligned}-\delta_\sigma W=\frac{1}{2(4\pi)^2}\int dx\,g^{1/2}\,\operatorname{tr}\Bigg[&-\tfrac{1}{6}(\Box\hat{P})\sigma-\tfrac{1}{180}(\Box R)\sigma\hat{1}-\tfrac{1}{12}\hat{\mathcal{R}}^2_{\mu\nu}\sigma\\&-\tfrac{1}{2}\hat{P}^2\sigma-\frac{1}{180}\left(1+2\frac{\Box_1}{\Box_2}-4\frac{\Box_3}{\Box_1}+\frac{\Box_3^2}{\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\mu\nu}\sigma_3\hat{1}\\&-\frac{1}{45}\left(\frac{2}{\Box_1}-\frac{\Box_3}{\Box_1\Box_2}\right)\nabla^\mu R_1^{\nu\lambda}\nabla_\nu R_{2\mu\lambda}\sigma_3\hat{1}\\&-\frac{1}{45}\frac{1}{\Box_1\Box_2}\nabla_\alpha\nabla_\beta R_{1\mu\nu}\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\sigma_3\hat{1}\Bigg]+O(\Re^3),\end{aligned} \tag{3.11}$$

where the notation in the nonlocal terms is the same as before, with σ playing the role of the third curvature. It is the latter equation that will be checked below by a direct calculation with W in (2.6).

We begin this check with calculating the result of the transformation (3.7) in the quadratic terms of W. For the quadratic terms of (2.6) we have

$$\begin{aligned}&\delta_\sigma\int dx\,g^{1/2}\,\operatorname{tr}\left(\sum_{i=1}^{5}\gamma_i(-\Box_2)\Re_1\Re_2(i)\right)\\&=\int dx\,g^{1/2}\,\operatorname{tr}\Big\{-\tfrac{3}{18}\hat{P}\Box\sigma+\tfrac{1}{30}R^{\mu\nu}\left[\gamma(-\Box)+\tfrac{16}{15}\right]\left(\nabla_\mu\nabla_\nu\sigma+\tfrac{1}{2}g_{\mu\nu}\Box\sigma\right)\hat{1}\\&\quad-\tfrac{3}{90}R\left[\gamma(-\Box)+\tfrac{37}{30}\right]\Box\sigma\hat{1}+\tfrac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\left[\delta_\sigma\gamma(-\Box)\right]\hat{\mathcal{R}}^{\mu\nu}+\tfrac{1}{2}\hat{P}\left[\delta_\sigma\gamma(-\Box)\right]\hat{P}\\&\quad+\tfrac{1}{60}R_{\mu\nu}\left[\delta_\sigma\gamma(-\Box)\right]R^{\mu\nu}\hat{1}-\tfrac{1}{180}R\left[\delta_\sigma\gamma(-\Box)\right]R\hat{1}\Big\},\end{aligned} \tag{3.12}$$

where

$$\gamma(-\Box)=-\ln\left(-\frac{\Box}{\mu^2}\right). \tag{3.13}$$

In the term linear in $R^{\mu\nu}$, for being able to use the Bianchi identity, one must commute $\gamma(-\Box)$ with $\nabla_\mu\nabla_\nu$, and the commutator cannot be neglected. As a result of this commutation, the linear nonlocal terms cancel, and we obtain

$$\begin{aligned}
&\delta_\sigma\int dx\, g^{1/2}\,\mathrm{tr}\left(\sum_{i=1}^{5}\gamma_i(-\Box_2)\Re_1\Re_2(i)\right)\\
&\quad=\int dx\, g^{1/2}\,\mathrm{tr}\Big\{-\tfrac{1}{6}(\Box\hat{P})\sigma-\tfrac{1}{180}(\Box R)\sigma\hat{1}\\
&\qquad+\tfrac{1}{30}R_{\mu\nu}[\gamma(-\Box),\nabla^\mu\nabla^\nu]\sigma\hat{1}\\
&\qquad+\tfrac{1}{12}\hat{\mathscr{R}}_{\mu\nu}[\delta_\sigma\gamma(-\Box)]\hat{\mathscr{R}}^{\mu\nu}+\tfrac{1}{2}\hat{P}[\delta_\sigma\gamma(-\Box)]\hat{P}\\
&\qquad+\tfrac{1}{60}R_{\mu\nu}[\delta_\sigma\gamma(-\Box)]R^{\mu\nu}\hat{1}-\tfrac{1}{180}R[\delta_\sigma\gamma(-\Box)]R\hat{1}\Big\},
\end{aligned}\tag{3.14}$$

where the first two terms correctly reproduce the linear contributions to the anomaly [10], and the remaining terms are already quadratic in the curvature.

For the calculation of the quadratic terms in (3.14) we use the spectral representation

$$\gamma(-\Box)=\int_0^\infty dm^2\left(\frac{1}{m^2-\Box}-\frac{1}{m^2+\mu^2}\right)\tag{3.15}$$

and the commutation (variation) rule for the inverse operator

$$\left[\frac{1}{m^2-\Box},\nabla^\mu\nabla^\nu\right]=-\frac{1}{m^2-\Box}[-\Box,\nabla^\mu\nabla^\nu]\frac{1}{m^2-\Box},\tag{3.16}$$

$$\delta_\sigma\frac{1}{m^2-\Box}=-\frac{1}{m^2-\Box}\delta_\sigma(-\Box)\frac{1}{m^2-\Box}\tag{3.17}$$

where, within the required accuracy, the factors on the right-hand sides can already be commuted freely. Doing the spectral-mass integral then gives

$$\begin{aligned}
&\int dx\, g^{1/2}R_{\mu\nu}[\gamma(-\Box),\nabla^\mu\nabla^\nu]\sigma\\
&\quad=-\int dx\, g^{1/2}\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3}[\Box_3,\nabla_3^\mu\nabla_3^\nu]R_{1\mu\nu}\sigma_3+\mathrm{O}(R^3_{..}),
\end{aligned}\tag{3.18}$$

and, similarly,

$$\begin{aligned}
&\int dx\, g^{1/2}\,\mathrm{tr}\{\Re_1[\delta_\sigma\gamma(-\Box_2)]\Re_2\}\\
&\quad=-\int dx\, g^{1/2}\,\mathrm{tr}\left(\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2}(\delta_\sigma\Box_2)\Re_1\Re_2\right)+\mathrm{O}(\Re^3).
\end{aligned}\tag{3.19}$$

There remain to be used in (3.19) the transformation laws (3.7), and in (3.18) the expression for the commutator

$$[\Box,\nabla_\mu\nabla_\nu]\sigma = 2\nabla_{(\mu}R_{\nu)\alpha}\nabla^\alpha\sigma + 2R_{\alpha(\mu}\nabla_{\nu)}\nabla^\alpha\sigma - \nabla_\alpha R_{\mu\nu}\nabla^\alpha\sigma - 2R_{\alpha\nu\beta\mu}\nabla^\alpha\nabla^\beta\sigma, \quad (3.20)$$

in which the Riemann tensor should be expressed through the Ricci tensor by Eq. (2.3).

The final result for (3.14) is

$$\delta_\sigma\int dx\,g^{1/2}\,\mathrm{tr}\left(\sum_{i=1}^{5}\gamma_i(-\Box_2)\Re_1\Re_2(i)\right) = \int dx\,g^{1/2}\,\mathrm{tr}\left(-\frac{1}{6}(\Box\hat P)\sigma - \frac{1}{180}(\Box R)\sigma\hat 1 + \sum_{i=1}^{10}M_i(\Box_1,\Box_2,\Box_3)\Re_1\Re_2\sigma_3(i)\right) + \mathrm{O}(\Re^3), \quad (3.21)$$

where $\Re_1\Re_2\sigma_3(i)$ are the following ten tensor structures:

$$\begin{aligned}
\Re_1\Re_2\sigma_3(1) &= R_1R_2\sigma_3\hat 1,\\
\Re_1\Re_2\sigma_3(2) &= R_1^{\mu\nu}R_{2\mu\nu}\sigma_3\hat 1,\\
\Re_1\Re_2\sigma_3(3) &= R_1^{\mu\nu}\nabla_\nu\nabla_\mu R_2\,\sigma_3\hat 1,\\
\Re_1\Re_2\sigma_3(4) &= \nabla^\mu R_1^{\nu\lambda}\,\nabla_\nu R_{2\mu\lambda}\,\sigma_3\hat 1,\\
\Re_1\Re_2\sigma_3(5) &= \nabla_\alpha\nabla_\beta R_{1\mu\nu}\,\nabla^\mu\nabla^\nu R_2^{\alpha\beta}\,\sigma_3\hat 1,\\
\Re_1\Re_2\sigma_3(6) &= \hat P_1R_2\sigma_3,\\
\Re_1\Re_2\sigma_3(7) &= \nabla_\alpha\nabla_\beta\hat P_1\,R_2^{\alpha\beta}\sigma_3,\\
\Re_1\Re_2\sigma_3(8) &= \hat P_1\hat P_2\sigma_3,\\
\Re_1\Re_2\sigma_3(9) &= \hat{\mathcal R}_1^{\mu\nu}\hat{\mathcal R}_{2\mu\nu}\sigma_3,\\
\Re_1\Re_2\sigma_3(10) &= \nabla_\mu\hat{\mathcal R}_1^{\mu\alpha}\,\nabla^\nu\hat{\mathcal R}_{2\nu\alpha}\sigma_3,
\end{aligned} \quad (3.22)$$

and for the form factors $M_i(\Box_1,\Box_2,\Box_3)$ we obtain

$$M_1 = \frac{-2\Box_1+5\Box_3}{720}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2} + \frac{\Box_1+\Box_2-\Box_3}{120}\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3},$$

$$M_2 = \frac{-2\Box_1-\Box_3}{120}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2} + \frac{(\Box_1-\Box_2-\Box_3)(\Box_1-\Box_3)}{60\Box_2} \times\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3},$$

$$
\begin{aligned}
M_3 &= -\frac{1}{30}\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3} - \frac{1}{30}\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3},\\
M_4 &= -\frac{1}{30}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2} + \frac{\Box_1-\Box_2-\Box_3}{15\Box_2}\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3},\\
M_5 &= \frac{1}{15\Box_2}\frac{\ln(\Box_1/\Box_3)}{\Box_1-\Box_3}, \quad M_6 = 0, \quad M_7 = 0,\\
M_8 &= \frac{-2\Box_2-\Box_3}{4}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2}, \quad M_9 = -\frac{\Box_3}{12}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2},\\
M_{10} &= \frac{1}{6}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2}.
\end{aligned}
\tag{3.23}
$$

The conformal transformation in the cubic terms of the effective action (2.6) is easier to carry out because, within the required accuracy, only the curvatures in $\Re_1\Re_2\Re_3$ need be varied. The result is again a sum of contributions of the ten tensor structures (3.22):

$$
\begin{aligned}
&\delta_\sigma\int dx\, g^{1/2}\,\mathrm{tr}\left(\sum_{i=1}^{29}\Gamma_i(-\Box_1,-\Box_2,-\Box_3)\Re_1\Re_2\Re_3(i)\right)\\
&\quad = \int dx\, g^{1/2}\,\mathrm{tr}\left(\sum_{i=1}^{10}N_i(\Box_1,\Box_2,\Box_3)\Re_1\Re_2\sigma_3(i)\right) + \mathrm{O}(\Re^3),
\end{aligned}
\tag{3.24}
$$

where the form factors $N_i(\Box_1, \Box_2, \Box_3)$ are the following combinations of the form factors $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$:

$$
\begin{aligned}
N_1 &= (-\Box_1+2\Box_3)\Gamma_{11}|_{\Box_1\leftrightarrow\Box_3}\\
&\quad + \tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3)+\Box_3(-\Box_1-\Box_2+\Box_3)]\\
&\quad \times\Gamma_{22}|_{\Box_1\leftrightarrow\Box_3} + \left[-\tfrac{1}{4}(\Box_1-\Box_2-\Box_3)^2 - \tfrac{1}{2}\Box_2\Box_3 + \tfrac{1}{4}(\Box_1-\Box_2-\Box_3)\right.\\
&\quad \left.\times(-\Box_1-\Box_2+\Box_3) - \tfrac{1}{4}\Box_3(-\Box_1-\Box_2+\Box_3)\right]\Gamma_{23}|_{\Box_1\leftrightarrow\Box_3}\\
&\quad + \tfrac{1}{16}(-\Box_1-\Box_2+\Box_3)^2\Gamma_{25}|_{\Box_1\leftrightarrow\Box_3} + \tfrac{1}{2}(-\Box_1+\Box_3)\\
&\quad \times\left[\tfrac{1}{2}(\Box_1-\Box_2-\Box_3)^2+\Box_2\Box_3\right]\Gamma_{27}|_{\Box_1\leftrightarrow\Box_3} + \tfrac{1}{16}(-\Box_1+\Box_2-\Box_3)\\
&\quad \times(-\Box_1-\Box_2+\Box_3)^2\Gamma_{28}|_{\Box_1\leftrightarrow\Box_3} + 9\Box_3\Gamma_9 + \tfrac{3}{8}(-\Box_1-\Box_2+\Box_3)\Gamma_{10}\\
&\quad - \tfrac{3}{2}\Box_3(-\Box_1-\Box_2+\Box_3)\Gamma_{22} + \tfrac{1}{4}(\Box_1-2\Box_3)(-\Box_1-\Box_2+\Box_3)\Gamma_{24}\\
&\quad + \tfrac{1}{8}(\Box_1-\Box_2-\Box_3)(-\Box_1-\Box_2+\Box_3)\Gamma_{25} + \tfrac{1}{32}(-\Box_1-\Box_2+\Box_3)\\
&\quad \times[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3)+\Box_3(-\Box_1-\Box_2+\Box_3)]\Gamma_{28},\\
N_2 &= \tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3)+\Box_3(-\Box_1-\Box_2+\Box_3)]\\
&\quad \times\Gamma_{24}|_{\Box_1\leftrightarrow\Box_3} + \tfrac{3}{2}\Box_3\Gamma_{10} + 3\Box_3\Gamma_{11},
\end{aligned}
$$

$$
\begin{aligned}
N_3 = & -\tfrac{1}{2}(-\Box_1-\Box_2+\Box_3)\Gamma_{24}|_{\Box_1\leftrightarrow\Box_2}+\tfrac{1}{2}(-\Box_1+\Box_2-\Box_3)\Gamma_{25}|_{\Box_1\leftrightarrow\Box_2} \\
& +\tfrac{1}{2}(-\Box_1-\Box_2+\Box_3)\Gamma_{25}|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{4}(-\Box_1+\Box_2-\Box_3) \\
& \times(-\Box_1-\Box_2+\Box_3)\Gamma_{28}|_{\Box_1\leftrightarrow\Box_3}+2\Gamma_{11}|_{\Box_2\leftrightarrow\Box_3}+(-\Box_1+\Box_2-\Box_3) \\
& \times\Gamma_{23}|_{\Box_2\leftrightarrow\Box_3}+\left[\tfrac{1}{2}(-\Box_1+\Box_2-\Box_3)^2+\Box_1\Box_3\right]\Gamma_{27}|_{\Box_2\leftrightarrow\Box_3} \\
& +\tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1-\Box_2+\Box_3)-\Box_3(-\Box_1-\Box_2+\Box_3)] \\
& \times\Gamma_{28}|_{\Box_2\leftrightarrow\Box_3}+3\Gamma_{10}-6\Box_3\Gamma_{22}+(\Box_1-2\Box_3)\Gamma_{24} \\
& +\tfrac{1}{2}(\Box_1-\Box_2-2\Box_3)\Gamma_{25}+\tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3) \\
& +\Box_3(-\Box_1-\Box_2+\Box_3)]\Gamma_{28} \\
& +\tfrac{3}{8}(-\Box_1+\Box_3) \\
& \times\left[(-\Box_1+\Box_2-\Box_3)^2+2\Box_1\Box_3\right]\Gamma_{29}, \\
N_4 = & \tfrac{1}{2}\Box_3\Gamma_{25}|_{\Box_1\leftrightarrow\Box_3}+3\Gamma_{10}+3\Box_3\Gamma_{23}+(\Box_1-\Box_2-2\Box_3)\Gamma_{25} \\
& +\tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3)\Box_3(-\Box_1-\Box_2+\Box_3)]\Gamma_{28}, \\
N_5 = & \Gamma_{25}|_{\Box_1\leftrightarrow\Box_3}-(\Box_1-\Box_2+2\Box_3)\Gamma_{28}|_{\Box_1\leftrightarrow\Box_3}-2\Gamma_{24}+3\Box_3\Gamma_{27} \\
& +\tfrac{3}{4}\left[(-\Box_1+\Box_2-\Box_3)^2+2\Box_1\Box_3\right]\Gamma_{29}, \\
N_6 = & -\tfrac{3}{4}\Box_3(-\Box_1-\Box_2+\Box_3)\Gamma_{15}|_{\Box_1\to\Box_2,\Box_2\to\Box_3,\Box_3\to\Box_1}+6\Box_3\Gamma_4|_{\Box_1\leftrightarrow\Box_3} \\
& +(-\Box_1+2\Box_3)\Gamma_5|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3) \\
& +\Box_3(-\Box_1-\Box_2+\Box_3)]\Gamma_{15}|_{\Box_1\leftrightarrow\Box_3}+\left[\tfrac{1}{4}(\Box_1-\Box_2-\Box_3)\Box_3\right. \\
& \left.+\tfrac{1}{2}(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_3)\right]\Gamma_{16}|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{2}(-\Box_1+\Box_3) \\
& \times\left[\tfrac{1}{2}(\Box_1-\Box_2-\Box_3)^2+\Box_2\Box_3\right]\Gamma_{26}|_{\Box_1\leftrightarrow\Box_3}, \\
N_7 = & -3\Box_3\Gamma_{15}|_{\Box_1\to\Box_2,\Box_2\to\Box_3,\Box_3\to\Box_1}+2\Gamma_5|_{\Box_1\leftrightarrow\Box_3}+(\Box_1-\Box_2-\Box_3) \\
& \times\Gamma_{16}|_{\Box_1\leftrightarrow\Box_3}+\left[\tfrac{1}{2}(\Box_1-\Box_2-\Box_3)^2+\Box_2\Box_3\right]\Gamma_{26}|_{\Box_1\leftrightarrow\Box_3}, \\
N_8 = & \left[\tfrac{1}{4}(\Box_1-\Box_2-\Box_3)^2+\tfrac{1}{2}\Box_2\Box_3\right]\Gamma_{17}|_{\Box_1\leftrightarrow\Box_3}+3\Box_3\Gamma_6, \\
N_9 = & 3\Box_3\Gamma_7|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{4}(\Box_1+\Box_2+3\Box_3)\Gamma_8|_{\Box_1\leftrightarrow\Box_3}-\tfrac{1}{2}\Box_1\Box_2\Gamma_{18}|_{\Box_1\leftrightarrow\Box_3} \\
& +\tfrac{1}{4}[(\Box_1-\Box_2-\Box_3)(-\Box_1+\Box_2-\Box_3)+\Box_3(-\Box_1-\Box_2+\Box_3)] \\
& \times\Gamma_{19}|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{4}[\Box_2(\Box_1-\Box_2-\Box_3)-\Box_2\Box_3]\Gamma_{21}|_{\Box_1\leftrightarrow\Box_3}, \\
N_{10} = & \Gamma_8|_{\Box_1\leftrightarrow\Box_3}+\tfrac{1}{2}(-\Box_1-\Box_2+2\Box_3)\Gamma_{18}|_{\Box_1\leftrightarrow\Box_3}+3\Box_3\Gamma_{20}|_{\Box_1\leftrightarrow\Box_3} \\
& +\tfrac{1}{2}(\Box_1-\Box_2-\Box_3)\Gamma_{21}|_{\Box_1\leftrightarrow\Box_3}
\end{aligned}
\tag{3.25}
$$

with $\Gamma_{11}|_{\Box_1\leftrightarrow\Box_3}$, etc. denoting the obvious permutations of the box operator arguments of $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$.

The total result of (3.21) and (3.24) is the following conformal variation of the effective action (2.6):

$$-\delta_\sigma W = \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}\,\mathrm{tr}\Bigg(-\tfrac{1}{6}(\Box\hat{P})\sigma - \tfrac{1}{180}(\Box R)\sigma\hat{1} + \sum_{i=1}^{10}(M_i+N_i)\Re_1\Re_2\sigma_3(i)\Bigg) + \mathrm{O}(\Re^3), \tag{3.26}$$

where the quadratic terms are determined by the sum $M_i + N_i$. There remain to be calculated the linear combinations of the third-order form factors Γ_i in (3.25). The most straightforward way of calculating these linear combinations is using the explicit expressions for Γ_i given in [24]. It is gratifying to observe that all terms with the basic third-order form factor $\Gamma(-\Box_1, -\Box_2, -\Box_3)$ cancel in the combinations N_i, all terms with the second-order form factors $\ln(\Box_n/\Box_m)$ cancel in the combinations $N_i + M_i$, and there remain only trees:

$$\begin{aligned}
&\tfrac{1}{2}\big(M_1+N_1+M_1|_{\Box_1\leftrightarrow\Box_2}+N_1|_{\Box_1\leftrightarrow\Box_2}\big)=0,\\
&\tfrac{1}{2}\big(M_2+N_2+M_2|_{\Box_1\leftrightarrow\Box_2}+N_2|_{\Box_1\leftrightarrow\Box_2}\big)\\
&\quad=-\frac{1}{180}-\frac{\Box_1}{180\Box_2}-\frac{\Box_2}{180\Box_1}+\frac{\Box_3}{90\Box_1}+\frac{\Box_3}{90\Box_2}-\frac{\Box_3^2}{180\Box_1\Box_2},\\
&M_3+N_3=0,\\
&\tfrac{1}{2}\big(M_4+N_4+M_4|_{\Box_1\leftrightarrow\Box_2}+N_4|_{\Box_1\leftrightarrow\Box_2}\big)=-\frac{1}{45\Box_1}-\frac{1}{45\Box_2}+\frac{\Box_3}{45\Box_1\Box_2},\\
&\tfrac{1}{2}\big(M_5+N_5+M_5|_{\Box_1\leftrightarrow\Box_2}+N_5|_{\Box_1\leftrightarrow\Box_2}\big)=-\frac{1}{45\Box_1\Box_2},\\
&M_6+N_6=0,\quad M_7+N_7=0,\\
&\tfrac{1}{2}\big(M_8+N_8+M_8|_{\Box_1\leftrightarrow\Box_2}+N_8|_{\Box_1\leftrightarrow\Box_2}\big)=-\tfrac{1}{2},\\
&\tfrac{1}{2}\big(M_9+N_9+M_9|_{\Box_1\leftrightarrow\Box_2}+N_9|_{\Box_1\leftrightarrow\Box_2}\big)=-\tfrac{1}{12},\\
&\tfrac{1}{2}\big(M_{10}+N_{10}+M_{10}|_{\Box_1\leftrightarrow\Box_2}+N_{10}|_{\Box_1\leftrightarrow\Box_2}\big)=0.
\end{aligned}\tag{3.27}$$

The symmetrizations on the left-hand sides of these equations correspond to the symmetries of the tensor structures (3.22). With the form factors (3.27), Eq. (3.26) takes precisely the form of the trace anomaly (3.11).

From the point of view of the expectation-value equations, one should also be able to carry out the calculation of the anomaly in terms of the spectral originals. In the derivation above this problem appears only with respect to the linear combinations (3.25) of the third-order form factors. Since the coefficients of these linear combinations contain $\Box$'s, a calculation in terms of integral originals encounters difficulties. However, the generalized spectral representation based on Eqs. (2.31), (2.32) makes the calculation feasible. The respective technique is worked up in the report [24]. The amount of calculations with the spectral forms is much smaller than with the explicit forms of the functions Γ_i, and the result is again Eq. (3.27).

In the calculation above, one can trace the two types of anomalies discussed in [13]. Eq. (3.21) is the contribution to the anomaly stemming from the scale-dependent logs in the quadratic terms of the effective action, and Eq. (3.24) is the contribution coming from varying the curvature in the finite cubic terms. Both contributions reduce, however, to the *scale-independent* logs in Eq. (3.23). These nonlocal logs cancel *only in the sum of the two contributions* in Eq. (3.26) leaving the trees (3.27). Trees were the only type of nonlocal form factors used in the constructions of the previous works [23,9,11] whereas in fact here we have to do with functions of three independent arguments $\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$, and no dimensional considerations can fix the dependence of these functions on the ratios $\Box_n/\Box_m$.

4. Heat kernel and the trace anomaly

The trace anomaly can also be derived by making the conformal transformation in the heat kernel which contributes to the effective action (2.5) via Eq. (2.13). To enable a comparison with the effective action (2.6), one must subtract from the heat kernel (2.14) the terms of zeroth and first order in the curvature (see a remark to Eq. (2.6)). For $\mathrm{Tr}\,K(s)$ with the lowest-order terms subtracted we introduce the notation

$$\mathrm{Tr}\,K'(s) = \mathrm{Tr}\,K(s) - \frac{1}{(4\pi s)^2}\int dx\, g^{1/2}\,\mathrm{tr}(\hat{1} + s\hat{P}). \tag{4.1}$$

The second-order terms in $\mathrm{Tr}\,K(s)$ transform like in (3.14) but, instead of $\gamma(-\Box)$, one has to deal with the form factors

$$f(-s\Box), \quad \frac{f(-s\Box)-1}{s\Box}, \quad \frac{f(-s\Box)-1-\frac{1}{6}s\Box}{(s\Box)^2}, \tag{4.2}$$

the form factors $f_i(-s\Box)$, $i = 1, \ldots, 5$, in (2.14) being the linear combinations of (4.2) with numerical coefficients [10,24]. Here

$$f(\xi) = \int_{\alpha \geqslant 0} d^2\alpha\, \delta(1-\alpha_1-\alpha_2)\, \exp(-\alpha_1\alpha_2\xi) = \int_0^1 d\alpha\, e^{-\alpha(1-\alpha)\xi} \tag{4.3}$$

is the basic second-order form factor in the trace of the heat kernel. The counterparts of Eqs. (3.18) and (3.19) are in this case

$$\int \mathrm{d}x\, g^{1/2} R_{\mu\nu}[f(-s\,\Box),\, \nabla^\mu\nabla^\nu]\sigma$$
$$= \int \mathrm{d}x\, g^{1/2} \frac{f(-s\,\Box_1) - f(-s\,\Box_3)}{\Box_1 - \Box_3}[\Box_3,\, \nabla_3^\mu \nabla_3^\nu] R_{1\mu\nu}\sigma_3 + \mathrm{O}(R^3_{..}), \tag{4.4}$$

$$\int \mathrm{d}x\, g^{1/2}\, \mathrm{tr}\{\Re_1[\delta_\sigma f(-s\,\Box_2)]\Re_2\}$$
$$= \int \mathrm{d}x\, g^{1/2}\, \mathrm{tr}\left(\frac{f(-s\,\Box_1) - f(-s\,\Box_2)}{\Box_1 - \Box_2}(\delta_\sigma\,\Box_2)\Re_1\Re_2\right) + \mathrm{O}(\Re^3). \tag{4.5}$$

The third-order terms in $\mathrm{Tr}\, K(s)$ transform in a way completely similar to the above [24]. An important distinction from the previous case is that the linear nonlocal terms do not cancel.

The total result for $\mathrm{Tr}\, K'(s)$ divided by s is of the form

$$\frac{1}{s}\delta_\sigma \mathrm{Tr}\, K'(s) = \frac{1}{(4\pi)^2}\int \mathrm{d}x\, g^{1/2}\, \mathrm{tr}\Bigg(\sigma\,\Box\, t_1(s,\Box)\hat{P} + \sigma\,\Box\, t_2(s,\Box)R\hat{1}$$
$$+ \sum_{i=1}^{10} T_i(s,\, \Box_1,\, \Box_2,\, \Box_3)\Re_1\Re_2\sigma_3(i)\Bigg) + \mathrm{O}(\Re^3), \tag{4.6}$$

where $\Re_1\Re_2\sigma_3(i)$ are the tensor structures (3.22), and the functions t_1, t_2, T_i are obtained as certain combinations of the form factors in the heat kernel. The differential equations for these form factors [15] can next be used the same way as in two dimensions [15] to bring the functions t_1, t_2, T_i to the form of total derivatives,

$$t_1 = \frac{\mathrm{d}}{\mathrm{d}s}\tilde{t}_1, \quad t_2 = \frac{\mathrm{d}}{\mathrm{d}s}\tilde{t}_2, \quad T_i = \frac{\mathrm{d}}{\mathrm{d}s}\tilde{T}_i. \tag{4.7}$$

Here

$$\tilde{t}_1 = \frac{f(-s\,\Box) - 1}{s\,\Box}, \tag{4.8}$$

$$\tilde{t}_2 = \frac{1}{12}\frac{f(-s\,\Box) - 1}{s\,\Box} - \frac{1}{2}\frac{f(-s\,\Box) - 1 - \frac{1}{6}s\,\Box}{(s\,\Box)^2}, \tag{4.9}$$

and the full results for $\tilde{T}_i$ can be found in the report [24]. Owing to (4.7), we need only the asymptotic behaviours of the form factors at large and small s of [15]. Indeed, the conformal variation of the effective action (2.6) is now obtained as

$$-\delta_\sigma W = \frac{1}{2}\int_0^\infty \frac{ds}{s}\delta_\sigma \operatorname{Tr} K'(s). \tag{4.10}$$

From (4.6) and (4.7) we find

$$-\delta_\sigma W = -\frac{1}{2(4\pi)^2}\int dx\, g^{1/2} \operatorname{tr}\Bigg(\sigma \Box \tilde{t}_1(0, \Box)\hat{P} + \sigma \Box \tilde{t}_2(0, \Box) R\hat{1} + \sum_{i=1}^{10} \tilde{T}_i(0, \Box_1, \Box_2, \Box_3)\Re_1\Re_2\sigma_3(i)\Bigg) + \mathrm{O}(\Re^3), \tag{4.11}$$

where use is made of the asymptotic behaviours at large s [15] to conclude that the functions $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{T}_i$ vanish at $s \to \infty$. By taking from [15] the asymptotic behaviours at small s, we obtain

$$\begin{aligned}
&\tilde{t}_1(0, \Box) = \tfrac{1}{6}, \quad \tilde{t}_2(0, \Box) = \tfrac{1}{180},\\
&\tfrac{1}{2}\left(\tilde{T}_1 + \tilde{T}_1|_{\Box_1 \leftrightarrow \Box_2}\right) = 0, \quad s = 0,\\
&\tfrac{1}{2}\left(\tilde{T}_2 + \tilde{T}_2|_{\Box_1 \leftrightarrow \Box_2}\right) = \frac{1}{180} + \frac{\Box_1}{180\Box_2} + \frac{\Box_2}{180\Box_1} - \frac{\Box_3}{90\Box_1} - \frac{\Box_3}{90\Box_2}\\
&\quad + \frac{\Box_3^{\,2}}{180\Box_1\Box_2}, \quad s = 0,\\
&\tilde{T}_3 = 0, \quad \tfrac{1}{2}\left(\tilde{T}_4 + \tilde{T}_4|_{\Box_1 \leftrightarrow \Box_2}\right) = \frac{1}{45\Box_1} + \frac{1}{45\Box_2} - \frac{\Box_3}{45\Box_1\Box_2}, \quad s = 0,\\
&\tfrac{1}{2}\left(\tilde{T}_5 + \tilde{T}_5|_{\Box_1 \leftrightarrow \Box_2}\right) = \frac{1}{45\Box_1\Box_2}, \quad \tilde{T}_6 = 0, \quad \tilde{T}_7 = 0, \quad s = 0,\\
&\tfrac{1}{2}\left(\tilde{T}_8 + \tilde{T}_8|_{\Box_1 \leftrightarrow \Box_2}\right) = \tfrac{1}{2}, \quad \tfrac{1}{2}\left(\tilde{T}_9 + \tilde{T}_9|_{\Box_1 \leftrightarrow \Box_2}\right) = \tfrac{1}{12}, \quad s = 0,\\
&\tfrac{1}{2}\left(\tilde{T}_{10} + \tilde{T}_{10}|_{\Box_1 \leftrightarrow \Box_2}\right) = 0, \quad s = 0.
\end{aligned} \tag{4.12}$$

With these expressions inserted in (4.11), one arrives at Eq. (3.11) which is the correct trace anomaly.

Because the conformal transformation is inhomogeneous in the curvature, the expansion in powers of the curvature does not preserve the exact conformal properties of the effective action. These properties can only be recovered order by order. An explicit calculation is required to check that higher-order terms in the curvature do not contribute to the anomaly, apart from the contributions assumed in Eq. (3.11). For the anomaly in two dimensions, such an explicit check has been carried out to third order in the curvature [15,24]. One can try to remove this shortcoming of covariant perturbation theory by using the ideas of [4], but such an improvement is already beyond the scope of the present paper.

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