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COVARIANT PERTURBATION THEORY
(III). Spectral representations of the third-order form factors

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Extension of covariant perturbation theory to third order in curvature requires the triple-spectral forms for one-loop vertex functions of massless fields to be established. For the simplest triangular graph with unit vertices the three spectral masses are known to form a triangle, and the spectral weight is $1/4\pi S$ where S is the area of this triangle. We generalize this result to an arbitrary derivative coupling.

1. Introduction

Covariant perturbation theory considered in the two preceding papers* [1,2] is a regular method for computing nonlocal terms in the effective action of quantum gauge fields. In this method the effective action is expanded in powers of a standard set of fields strengths characterizing any given field model. Symbolically this expansion is of the form

$$W=\int dx\, g^{1/2}\{\Re+F_2(\Box_1,\Box_2)\Re_1\Re_2+F_3(\Box_1,\Box_2,\Box_3)\Re_1\Re_2\Re_3+\mathrm{O}[\Re^4]\}, \quad (1.1)$$

where $dx\, g^{1/2}$ is the space-time volume element, and $\Re$ is the collective notation for a set of field strengths (curvatures, see refs. [1,2]) and their finite-order derivatives. The coefficients of the expansion, the nonlocal form factors F_2, F_3 etc., are functions of covariant d'alembertians $\Box$ only. The numbers 1, 2, 3 labeling the d'alambertians and curvatures in eq. (1.1) indicate that $\Box_1$ acts on $\Re_1 \equiv \Re(x_1)$, $\Box_2$ acts on $\Re_2 \equiv \Re(x_2)$, etc. with subsequent setting $x_1 = x_2 = \cdots = x$. The second-

* For more references and the discussion of relevant physical problems see paper II [2].

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order form factor is more conveniently introduced as a function of one argument:

$$\int \mathrm{d}x\, g^{1/2} F_2(\Box_1,\Box_2)\,\Re_1\Re_2 = \int \mathrm{d}x\, g^{1/2}\Re\, F_2(\Box)\,\Re\,, \tag{1.2}$$

$$F_2(\Box) = F_2(\Box,\Box)\,, \tag{1.3}$$

and beginning with fourth order in the curvature the d'alambertians act on products of $\Re$'s.

The effective action is actually computed for euclidean signature of space-time in which case $\Box$'s are the covariant laplacians. As discussed in ref. [1], two important quantum problems in lorentzian space-time boil down to the calculation of the euclidean effective action. These are (i) the problem about transition amplitudes between the standard in- and out-states defined in the remote past and remote future, and (ii) the problem about expectation values in an in- (or out-) state. (For the general setting of these problems see ref. [3].) In particular, the effective equations for the mean field which may be either the normalized matrix element between the in-vacuum and out-vacuum states or the expectation value in the in-vacuum (or out-vacuum) state can be obtained by varying the euclidean effective action. It turns out that the form factors in the effective equations for these two cases are given by one and the same functions $F_2(\Box_1,\Box_2)$, $F_3(\Box_1,\Box_2,\Box_3)$ etc. – the ones obtained by computing the euclidean effective equations, and the difference is only in the boundary conditions for the operator arguments $\Box$ of functions F. These boundary conditions are most easily formulated if the euclidean form factors can be represented as spectral integrals with respect to *all of their arguments*:

$$F_2(\Box) = \int \mathrm{d}m^2 \frac{\rho_2(m^2)}{m^2-\Box}\,, \tag{1.4}$$

$$F_3(\Box_1,\Box_2,\Box_3) = \int \mathrm{d}m_1^2\,\mathrm{d}m_2^2\,\mathrm{d}m_3^2 \frac{\rho_3(m_1^2,m_2^2,m_3^2)}{(m_1^2-\Box_1)(m_2^2-\Box_2)(m_3^2-\Box_3)}\,, \tag{1.5}$$

etc. so that all nonlocal operators can be expressed entirely in terms of Green functions

$$G = \frac{1}{m^2-\Box}\,.$$

Then the algorithm of obtaining the effective equations for the in–out or in–in (out–out) mean fields is as follows: compute the euclidean effective equations and go over to lorentzian signature, replacing all euclidean Green functions by the Feynman Green functions or retarded (advanced) Green functions respectively [4, 1]. In general, expressing the nonlocal form factors in terms of Green functions via spectral integrals seems the only way to make them tractable when dealing with the effective equations.

Spectral representations play, therefore, an important role in the effective action theory. For the one-loop effective action the precise form of expansion (1.1) to second order in the curvature is obtained in ref. [2]. In four dimensions the only nonlocal form factor which arises in this approximation is

$$F_2(\Box) = \ln(-\Box/\mu^2) \tag{1.6}$$

(μ^2 is a renormalization parameter) and its spectral representation is obvious:

$$\ln(-\Box/\mu^2) = \int_0^\infty \mathrm{d}m^2 \left(\frac{1}{m^2+\mu^2} - \frac{1}{m^2-\Box} \right), \quad \Box < 0. \tag{1.7}$$

Preliminary consideration of terms cubic in the curvature can be found in refs. [1,2]. The full cubic expression requires much work and we hope to present it in a separate publication*. A distinct part of this work is establishing the triple spectral forms (1.5) for all third-order form factors. This is the purpose of the present paper.

Spectral theory is, of course, a subject covered by a vast amount of literature. In particular, the single and double spectral forms for the three-point functions were repeatedly discussed in various contexts [5–12]. The triple spectral form was proposed in ref. [5] and, for the massless case, correctly established in ref. [8]. However, this earlier work was mainly confined to the consideration of simplest interactions whereas for the present purposes we need explicit formulae for an arbitrary derivative coupling. Such formulae are obtained below.

2. Triple spectral forms for the one-loop form factors

The simplest third-order euclidean form factor is given by a triangular graph (fig. 1) with unit vertices and massless scalar propagators. In four dimensions

$$\Gamma(x,y,z) = G(x,y)G(y,z)G(z,x), \tag{2.1}$$

$$G(x,y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \frac{\mathrm{e}^{ip(x-y)}}{p^2}. \tag{2.2}$$

* The computation of cubic terms is motivated by a desire to reproduce the conformal anomaly in four-dimensional gravity theory (see a discussion in ref. [2]).

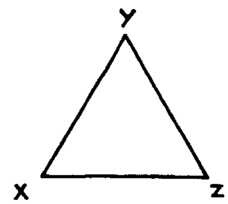

Fig. 1. The triangular graph contribution to the vertex function: $\Gamma(x, y, z)$.

With a convenient normalization of the form factor one has

$$\Gamma(y_1, y_2, y_3) = \frac{1}{(4\pi)^2} \int d^4x\, F(\Box_{x_1}, \Box_{x_2}, \Box_{x_3}) \times \delta(x_1 - y_1)\, \delta(x_2 - y_2)\, \delta(x_3 - y_3) \Big|_{x_1 = x_2 = x_3 = x} . \tag{2.3}$$

By introducing for each of the three propagators the proper-time integral

$$\frac{1}{p^2} = \int_0^\infty ds\, e^{-sp^2} \tag{2.4}$$

and carrying out the gaussian momentum integration one finds the form of the function F:

$$F(\Box_1, \Box_2, \Box_3) = \int_0^\infty \frac{ds_1\, ds_2\, ds_3}{(s_1 + s_2 + s_3)^2} \times \exp\left(\frac{s_2 s_3 \Box_1 + s_1 s_3 \Box_2 + s_1 s_2 \Box_3}{s_1 + s_2 + s_3} \right), \quad \Box_1 < 0, \Box_2 < 0, \Box_3 < 0. \tag{2.5}$$

A convenient starting expression for the form factor arises after introducing new variables

$$s = s_1 + s_2 + s_3, \qquad \alpha_1 = \frac{s_1}{s}, \qquad \alpha_2 = \frac{s_2}{s}, \qquad \alpha_3 = \frac{s_3}{s} \tag{2.6}$$

and carrying out the integration over s:

$$F(\Box_1, \Box_2, \Box_3) = \int d^3\alpha\, \delta\left(1 - \sum \alpha\right) \frac{1}{\Omega} . \tag{2.7}$$

Here

$$\Omega \equiv \alpha_2\alpha_3(-\Box_1) + \alpha_1\alpha_3(-\Box_2) + \alpha_1\alpha_2(-\Box_3), \tag{2.8}$$

and

$$\sum\alpha \equiv \alpha_1 + \alpha_2 + \alpha_3, \qquad \int d^3\alpha \equiv \int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\alpha_3 . \tag{2.9}$$

If the vertices of the diagram in fig. 1 contain derivatives, more complicated form factors arise. All of them are linear combinations of the integrals

$$\int d^3\alpha\,\delta\left(1 - \sum\alpha\right) \frac{\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}}{\Omega} \tag{2.10}$$

which generalize eq. (2.7). Moreover, since for theories with derivative coupling the triangular graph diverges, there are also form factors of the form

$$\int \mathrm{d}^3\alpha\,\delta\left(1 - \sum\alpha\right) \alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3} \ln(\Omega/\mu^2) \tag{2.11}$$

where μ^2 is a renormalization parameter. In eqs. (2.10) and (2.11), n_1, n_2, n_3 are integers.

Our goal is the triple spectral representation of functions (2.7), (2.10) and (2.11) in the variables $\Box_1$, $\Box_2$, $\Box_3$:

$$F(\Box_1,\Box_2,\Box_3) = \int_{\mathrm{V}} \mathrm{d}m_1^2\,\mathrm{d}m_2^2\,\mathrm{d}m_3^2 \frac{\rho\left(m_1^2, m_2^2, m_3^2\right)}{\left(m_1^2 - \Box_1\right)\left(m_2^2 - \Box_2\right)\left(m_3^2 - \Box_3\right)} \tag{2.12}$$

and similarly for eqs. (2.10) and (2.11), where V is some integration region, and $\rho(m_1^2, m_2^2, m_3^2)$ is an unknown spectral weight function.

The result for the basic form factor (2.7) is remarkably simple. The integration region V is a region of such values of the three masses

$$m_i \equiv \sqrt{m_i^2} \tag{2.13}$$

that m_i form a triangle (fig. 2), and the spectral weight equals

$$\rho\left(m_1^2, m_2^2, m_3^2\right) = \frac{1}{4\pi S}, \tag{2.14}$$

where S is the area of this triangle [8]. Explicitly:

$$\mathrm{V}: \qquad m_1 < m_2 + m_3, \qquad m_2 < m_1 + m_3, \qquad m_3 < m_1 + m_2 \tag{2.15}$$

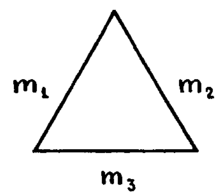

Fig. 2. The triangle of masses defining the spectral density for the triangular graph.

and

$$\rho\left(m_1^2, m_2^2, m_3^2\right)=\frac{1}{\pi} \frac{1}{\sqrt{\left(m_1+m_2+m_3\right)}} \times \frac{1}{\sqrt{\left(m_1+m_2-m_3\right)\left(m_1+m_3-m_2\right)\left(m_3+m_2-m_1\right)}} . \tag{2.16}$$

As regards the integrals (2.10) and (2.11), they can be reduced to the basic form factor (2.7). The reduction formulae read

$$\int \mathrm{d}^3\alpha\, \delta\left(1-\sum \alpha\right) \frac{\alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3}}{\Omega} = \frac{1}{\left(n_1+n_2+n_3\right)!}\left(-\square_1\right)^{n_1} \frac{\partial^{n_1}}{\partial \square_1^{n_1}}\left(-\square_2\right)^{n_2} \frac{\partial^{n_2}}{\partial \square_2^{n_2}}\left(-\square_3\right)^{n_3} \frac{\partial^{n_3}}{\partial \square_3^{n_3}} F\left(\square_1, \square_2, \square_3\right), \tag{2.17}$$

$$\int \mathrm{d}^3\alpha\, \delta\left(1-\sum \alpha\right) \ln \frac{\Omega}{\mu^2} = \tfrac{1}{6} \sum_{i=1}^{3} \ln\left(-\square_i / \mu^2\right)-\tfrac{3}{2}-\tfrac{1}{6} \sum_{i=1}^{3}\left(\square_i+\square_i^2 \frac{\partial}{\partial \square_i}\right) F\left(\square_1, \square_2, \square_3\right), \tag{2.18}$$

$$\int \mathrm{d}^3\alpha\, \delta\left(1-\sum \alpha\right) \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \ln\left(\Omega / \mu^2\right) = \frac{n_1! n_2! n_3!}{\left(n_1+n_2+n_3+2\right)!}\left(\sum_{l=1}^{n_1} \frac{1}{l}+\sum_{l=1}^{n_2} \frac{1}{l}+\sum_{l=1}^{n_3} \frac{1}{l}-2 \sum_{l=3}^{n_1+n_2+n_3+2} \frac{1}{l}\right)$$
$$-\frac{2}{\left(n_1+n_2+n_3+2\right)!}\left(-\square_1\right)^{n_1+1} \frac{\partial}{\partial \square_1^{n_1}}\left(-\square_2\right)^{n_2+1} \frac{\partial}{\partial \square_2^{n_2}}\left(-\square_3\right)^{n_3+1} \frac{\partial}{\partial \square_3^{n_3}} \times \frac{1}{\square_1 \square_2 \square_3} \int \mathrm{d}^3\alpha\, \delta\left(1-\sum \alpha\right) \ln\left(\Omega / \mu^2\right). \tag{2.19}$$

The spectral representations for the functions (2.17), (2.18) and (2.19) are obtained by inserting expressions (1.7) and (2.12) for the basic form factors, and using the identities

$$(-\Box)^n \frac{\partial^n}{\partial\Box^n}\frac{1}{m^2-\Box} = \frac{n!}{m^2-\Box}\left(\frac{-\Box}{m^2-\Box}\right)^n = \left(\frac{\mathrm{d}}{\mathrm{d}m^2}\right)^n \frac{(m^2)^n}{m^2-\Box}.$$

3. Derivation of the basic spectral formula

For completeness we begin the derivation with the basic spectral weight function (2.16). The elementary proof of this result [8] is as follows.

By making the replacement of the integration variables in (2.7):

$$\begin{gathered} \alpha_1 = \lambda z x, \qquad \alpha_2 = \lambda z(1-x), \qquad \alpha_3 = \lambda(1-z), \\ 0 \leqslant \lambda < \infty, \qquad 0 \leqslant z \leqslant 1, \qquad 0 \leqslant x \leqslant 1, \\ \frac{\partial(\alpha_1,\alpha_2,\alpha_3)}{\partial(\lambda,z,x)} = z\lambda^2 \end{gathered} \tag{3.1}$$

and doing the integral over λ with the aid of the delta function one obtains the following expression for the function $F(\Box_1,\Box_2,\Box_3)$:

$$F = \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \left[zx(1-x)(-\Box_3) + (1-x)(1-z)(-\Box_1) + x(1-z)(-\Box_2)\right]^{-1}, \tag{3.2}$$

where $\Box_i < 0$.

Note that the $\Box$'s in eq. (3.2) are already in the denominator, as required. Therefore, the simplest way to obtain the spectral representation is systematically introducing integrals over masses with the aid of delta functions and then using these delta functions to perform the integrals over the parameters. Thus, we write

$$F = \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}x \int_0^\infty \frac{\mathrm{d}m^2}{m^2-\Box_3} \times \delta\big((1-x)(1-z)(-\Box_1) + x(1-z)(-\Box_2) - m^2 zx(1-x)\big). \tag{3.3}$$

The integral over z can now be done. Since the only zero of the argument of the delta function in the variable z always lies in the integration region we obtain

$$F=\int_0^\infty \frac{\mathrm{d}m^2}{m^2-\Box_3}\int_0^1 \mathrm{d}x\left[x(1-x)m^2+x(-\Box_2)+(1-x)(-\Box_1)\right]^{-1}. \quad (3.4)$$

Next we repeat the trick:

$$F=\int_0^\infty \frac{\mathrm{d}m^2}{m^2-\Box_3}\int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}\mu^2\frac{1}{\mu^2-\Box_2}\delta\left(x(1-x)m^2+(1-x)(-\Box_1)-x\mu^2\right) \quad (3.5)$$

and use the delta function to perform the integral over x. The zeroes of the argument of the delta function in eq. (3.5) are

$$x_\pm=\frac{1}{2m^2}\left[-\left(\mu^2-m^2-\Box_1\right)\pm\sqrt{\left(\mu^2-m^2-\Box_1\right)^2-4m^2\Box_1}\,\right]. \quad (3.6)$$

With $\Box_1$ negative, both roots are real. As seen from eq. (3.6),

$$x_-<0, \quad x_+>0 \quad (3.7)$$

always. By rewriting eq. (3.6) as

$$2m^2(x_+-1)=\sqrt{\left(\mu^2+m^2-\Box_1\right)^2-4m^2\mu^2}-\left(\mu^2+m^2-\Box_1\right) \quad (3.8)$$

one makes sure that

$$x_+<1 \quad (3.9)$$

always. Thus x_- always lies outside and x_+ inside the integration region in x. As a result we obtain

$$F=\int_0^\infty \frac{\mathrm{d}m^2}{m^2-\Box_3}\int_0^\infty \frac{\mathrm{d}\mu^2}{\mu^2-\Box_2}\left[\left(\mu^2-m^2-\Box_1\right)^2-4m^2\Box_1\right]^{-1/2}. \quad (3.10)$$

To write the spectral representation for the remaining explicit function of $\Box_1$ we note that this function can be transformed as

$$\left[\left(\mu^2-m^2-\Box_1\right)^2-4m^2\Box_1\right]^{-1/2}=\left[\left(-\Box_1+(\mu+m)^2\right)\left(-\Box_1+(\mu-m)^2\right)\right]^{-1/2} \quad (3.11)$$

and use the identity

$$[(y+a)(y+b)]^{-1/2} = \frac{1}{\pi}\int_a^b \frac{dx}{x+y}[(x-a)(b-x)]^{-1/2} \tag{3.12}$$

with $y = -\Box_1 > 0$, $a = (\mu - m)^2$, $b = (\mu + m)^2$. The latter integral is elementary.

Thus we obtain the desired triple spectral representation of the form factor

$$F = \frac{1}{\pi}\int_0^\infty \frac{dm^2}{m^2 - \Box_3}\int_0^\infty \frac{d\mu^2}{\mu^2 - \Box_2}\int_{(\mu-m)^2}^{(\mu+m)^2} \frac{d\nu^2}{\nu^2 - \Box_1}$$

$$\times\left(\sqrt{\left(\nu^2 - (\mu - m)^2\right)\left((\mu + m)^2 - \nu^2\right)}\right)^{-1}. \tag{3.13}$$

The triple integral in eq. (3.13) can be rewritten as

$$\int_0^\infty dm^2 \int_0^\infty d\mu^2 \int_{(\mu-m)^2}^{(\mu+m)^2} d\nu^2 = \int_V dm^2\, d\mu^2\, d\nu^2, \tag{3.14}$$

where

$$\mathrm{V}:\ (\mu - m)^2 < \nu^2 < (\mu + m)^2. \tag{3.15}$$

Hence $(\mu + m - \nu) > 0$, and either

$$(\nu + \mu - m) > 0, \qquad (\nu + m - \mu) > 0 \tag{3.16a}$$

or

$$(\nu + \mu - m) < 0, \qquad (\nu + m - \mu) < 0. \tag{3.16b}$$

The latter possibility is excluded, because the sum of the two inequalities in (3.16b) gives $2\nu < 0$. Thus

$$\mathrm{V}: \quad (\mu + m - \nu) > 0, \qquad (\nu + \mu - m) > 0, \qquad (\nu + m - \mu) > 0, \tag{3.17}$$

that is m, μ and ν form a triangle. As for the square root in eq. (3.13), it is of the form

$$\sqrt{(\mu + m + \nu)(\mu + m - \nu)(\nu + \mu - m)(\nu + m - \mu)} \tag{3.18}$$

which up to the factor $1/4$ is the Heron formula for the area of this triangle!

The above spectral representation can also be derived by consecutively applying the Cauchy formula to the three arguments of the function (2.7). Thus, for $\Box_1$ and $\Box_2$ negative, the spectral weight in the variable $\Box_3$ is given by the jump across a cut along the non-negative real axis:

$$F(\Box_1,\Box_2,m^2+i\varepsilon)-F(\Box_1,\Box_2,m^2-i\varepsilon)=2i\pi\int d^3\alpha\,\delta\left(1-\sum\alpha\right)\delta(\Omega)\,. \quad (3.19)$$

After doing two integrals over the parameters one writes a similar expression for the spectral weight of the function (3.19) in the variable $\Box_2$. Finally, application of the Cauchy formula to the explicit function (3.11) gives eq. (3.12). The procedure amounts to solving the same equations as above and leads, of course, to the same result.

4. Derivation of the reduction formulae

To derive relations (2.17)–(2.19) it is useful to consider the function

$$\frac{1}{\Gamma(\varepsilon)}\int_0^\infty\frac{ds_1\,ds_2\,ds_3}{(s_1+s_2+s_3)^{3-\varepsilon}}\frac{s_1^{n_1}s_2^{n_2}s_3^{n_3}}{(s_1+s_2+s_3)^{n_1+n_2+n_3}}\exp\left(\frac{s_2s_3\Box_1+s_1s_3\Box_2+s_1s_2\Box_3}{s_1+s_2+s_3}\right)$$

$$=\int d^3\alpha\,\delta\left(1-\sum\alpha\right)\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\Omega^{-\varepsilon} \quad (4.1)$$

which is a generalization of eqs. (2.5) and (2.10). The integral (2.10) is (4.1) at $\varepsilon=1$. For the function (4.1) the following relation is true

$$\int d^3\alpha\,\delta\left(1-\sum\alpha\right)\alpha_1^{n_1+1}\alpha_2^{n_2}\alpha_3^{n_3}\Omega^{-\varepsilon}$$

$$=\frac{n_1+1-\varepsilon-\Box_1\,\partial/\partial\Box_1}{n_1+n_2+n_3+3-2\varepsilon}\int d^3\alpha\,\delta\left(1-\sum\alpha\right)\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\Omega^{-\varepsilon} \quad (4.2)$$

which makes it possible to reduce the power of the monomial in α. The proof of eq. (4.2) is as follows.

Let us make the following replacement of variables in the integral on the l.h.s. of eq. (4.1):

$$u_1=\frac{s_2s_3}{s_1+s_2+s_3},\qquad u_2=\frac{s_1s_3}{s_1+s_2+s_3},\qquad u_3=\frac{s_1s_2}{s_1+s_2+s_3},$$

$$\left|\frac{\partial s}{\partial u}\right|=\frac{(u_1u_2+u_2u_3+u_3u_1)^3}{(u_1u_2u_3)^2}\,. \quad (4.3)$$

Then eq. (4.1) takes the form

$$\int d^3\alpha\, \delta\Big(1-\sum\alpha\Big)\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\Omega^{-\varepsilon}$$

$$=\frac{1}{\Gamma(\varepsilon)}\int d^3u \frac{(u_1u_2u_3)^{1-\varepsilon}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}}{(u_1u_2+u_2u_3+u_3u_1)^{3-2\varepsilon}}\exp\Big(\sum_i u_i\Box_i\Big) \tag{4.4}$$

where the α's on the right-hand side are functions of u:

$$\alpha_1=\frac{u_2u_3}{u_1u_2+u_2u_3+u_3u_1},\qquad \alpha_2=\frac{u_1u_3}{u_1u_2+u_2u_3+u_3u_1},$$

$$\alpha_3=\frac{u_1u_2}{u_1u_2+u_2u_3+u_3u_1}, \tag{4.5}$$

which satisfy the equations

$$\frac{u_1}{\alpha_1}\frac{\partial\alpha_1}{\partial u_1}=\alpha_1-1,\qquad \frac{u_1}{\alpha_2}\frac{\partial\alpha_2}{\partial u_1}=\alpha_1,\qquad \frac{u_1}{\alpha_3}\frac{\partial\alpha_3}{\partial u_1}=\alpha_1. \tag{4.6}$$

The action of the operator

$$n_1+1-\varepsilon-\Box_1\frac{\partial}{\partial\Box_1}$$

on (4.4) boils down to its action on the exponential, which can be written as

$$\left(n_1+1-\varepsilon-u_1\frac{\partial}{\partial u_1}\right)\exp\Big(\sum_i u_i\Box_i\Big).$$

Now integrate over u_1 by parts using eqs. (4.6). The result will be eq. (4.2).

Repeated application of the recurrence relation (4.2) makes use of the identity

$$\left(n-1-\varepsilon-\Box\frac{\partial}{\partial\Box}\right)\dots\left(k-\varepsilon-\Box\frac{\partial}{\partial\Box}\right)$$

$$=(-\Box)^{n-\varepsilon}\frac{\partial^{n-k}}{\partial\Box^{n-k}}(-\Box)^{\varepsilon-k},\qquad k\leqslant n-1 \tag{4.7}$$

which can be proved by induction in n. In this way one obtains the reduction formula

$$\int d^3\alpha\, \delta\left(1-\sum\alpha\right)\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\Omega^{-\varepsilon}$$

$$=\frac{\Gamma(3-2\varepsilon)}{\Gamma(n_1+n_2+n_3+3-2\varepsilon)}(-\Box_1\Box_2\Box_3)^{1-\varepsilon}$$

$$\times(-\Box_1)^{n_1}\frac{\partial^{n_1}}{\partial\Box_1^{n_1}}(-\Box_2)^{n_2}\frac{\partial^{n_2}}{\partial\Box_2^{n_2}}(-\Box_3)^{n_3}\frac{\partial^{n_3}}{\partial\Box_3^{n_3}}$$

$$\times(-\Box_1\Box_2\Box_3)^{\varepsilon-1}\int d^3\alpha\, \delta\left(1-\sum\alpha\right)\Omega^{-\varepsilon} \tag{4.8}$$

which is a generalization of eq. (2.17).

In order to reduce the integral

$$\int d^3\alpha\, \delta\left(1-\sum\alpha\right)\ln(\Omega/\mu^2) \tag{4.9}$$

to the basic form factor we make the replacement (3.1) of integration variables in eq. (4.9) and integrate over z by parts. This gives

$$\int d^3\alpha\, \delta(1-\Sigma\alpha)\ln(\Omega/\mu^2)=\tfrac{1}{2}\ln(-\Box_3/\mu^2)-\tfrac{3}{2}-\tfrac{1}{2}\int d^3\alpha\, \delta\left(1-\sum\alpha\right)\frac{\alpha_2\Box_1+\alpha_1\Box_2}{\Omega}. \tag{4.10}$$

By symmetry, relation (4.10) remains valid under a cyclic permutation of indices on $\Box$'s and α's. This fact is made apparent by the identity

$$\ln(\Box_2/\Box_1)=\int d^3\alpha\, \delta\left(1-\sum\alpha\right)\frac{\alpha_3(\Box_1-\Box_2)+(\alpha_1-\alpha_2)\Box_3}{\Omega} \tag{4.11}$$

and similar identities with cyclically permuted indices. The identity (4.11) is proved by again using the parametrization (3.1) in which it takes the form

$$\int_0^\infty d\lambda\, \lambda^2\delta(1-\lambda)\int_0^1 dz\left[\ln\Omega|_{x=1}-\ln\Omega|_{x=0}-\int_0^1 dx\frac{1}{\Omega}\frac{\partial\Omega}{\partial x}\right]\equiv 0.$$

Explicit symmetrization of eq. (4.10) with subsequent use of eq. (2.17) gives eq. (2.18). Finally, the reduction formula (2.19) is obtained by differentiating eq. (4.8) with respect to ε and setting $\varepsilon=0$.

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