\[ \nabla_k E = \frac{\hbar^2}{m \omega^2} k \]

\[ D(E) = \frac{1}{\pi} \frac{m \omega^2}{\hbar^2} \quad \text{as before} \]

\[ D(E) = \frac{1}{2\pi} \sum_{E=E(k)} \frac{1}{|\nabla_k E(k)|} \]

\[ \nabla_k E(k) = \frac{\hbar^2}{m \omega} k \]

\[ k = \left( \frac{2mE}{\hbar^2} \right)^{1/2} \sqrt{E - E_{mn}} \]

\[ D(E) = \frac{1}{\pi} \left( \frac{\hbar^2}{m \omega^2} \right)^{3/2} (E - E_{mn})^{-1/2} \]

Distribution function

Now that we can count the states, we need to figure out if they are actually occupied and to what extent. For this purpose we will use the notion of a "distribution function" which tells us how energy is
actually distributed among the available energy levels. For example, the total energy in a level $E(k)$ would be the product of the energy of that level $E(k)$ times the degeneracy of that level $N(E)$, or, more commonly, the density of states of that level $D(E)$, and the actual number of particles having energy $E$, given by the distribution function $f(E)$.

How do we derive such a function? First, we will consider the equilibrium case, where we have a collection of energy levels $E_i$, with $g_i$ states per level. We fill them with $n_i$ particles so that the total number of particles is $N = E_i n_i$ and having a total energy $\sum E_i n_i$. 

\[
\begin{align*}
E_1 & \quad E_2 \\
E_i & \quad \sum E_i
\end{align*}
\]
Next we ask: how many ways are there to fill each level $E_i$?

This is the same as asking how many ways can we put $n_i$ objects into $g_i$ boxes.

There are two answers to this question: one in the case we can put as many particles in each box and another when we can only put 1 particle in each box.

Let's tackle the second problem first: if we can only put 1 particle in each box, then we have $g_i$ choices for the 1st particle. After we pick a box for the 1st particle, we have 0 choices for the 2nd, $g_i - 2$ for the 3rd, and so on until we choose boxes for all $n_i$ particles. This gives us

$$W(n_i, g_i) = \frac{g_i!}{(g_i - n_i)!} \text{ choices.}$$

We also have to note that
there can be two versions here. one is we can distinguish the particles, which gives us the same

\[ W(n_i, g_i) = \frac{g_i^!}{(g_i - n_i)!} \]

and the other is if the particles are indistinguishable, where we have to divide by \( g_i! \) to account for all the ways we can order

\[ W(n_i, g_i) = \frac{g_i^!}{(g_i - n_i)! n_i!} \]

Now the number of particles \( n_i \) can be any number, so the total number of ways of distributing them will be a product of ways of distributing \( 1, 2, 3 \ldots \) particles

\[ W = \prod_n W(n_i, g_i) \]

The entropy of the system can now be written as \( S = k_B \ln W \)

Equilibrium configuration will maximize entropy so we find the
equilibrium number of particles in each energy level is by maximizing $S = k_B \ln W$, which occurs when
\[\frac{\partial S}{\partial n_i} = 0,\]
under the additional constraint that the total number of particles is $N = \sum n_i$ and the total energy is $E = \sum E_i n_i$.

We introduce these constraints using Lagrange multipliers by adding two factors $\alpha$ and $\beta$, one for each constraint being satisfied,

\[
f(n_i) = k_B \ln W + \alpha (N - \sum n_i) + \beta (E - \sum E_i n_i)
\]

\[
\frac{\partial f}{\partial n_i} = k_B \ln \frac{n_i}{\langle g_i \rangle} + \alpha (N - \sum n_i) + \beta (E - \sum E_i n_i)
\]

\[
= k_B \sum_{i} \ln g_i - \ln \left[ \langle g_i \rangle \right] - \ln \left( \langle g_i \rangle \right)
\]

\加上 $\alpha (N - \sum n_i) + \beta (E - \sum E_i n_i)$

Using Stirling's Approximation
\[
\ln x! \approx x \ln x
\]

\[
= k_B \sum_{i} g_i \ln g_i - \langle g_i - n_i \rangle \ln \langle g_i - n_i \rangle - n_i \ln n_i - \frac{1}{2} \alpha (N - \sum n_i) + \beta (E - \sum E_i n_i)
\]
\[
\alpha = \frac{\partial f}{\partial n_i} = k_b \left[ \frac{g_i}{g_i - n_i} - \frac{n_i}{g_i - n_i} \right] - \ln (n_i) - 1 + \beta E_i \Rightarrow \alpha - \frac{\partial f}{\partial n_i} - \beta E_i = 0
\]

\[
\ln \left( \frac{g_i - n_i}{n_i} \right) = \frac{\alpha}{k_b} + \frac{\beta E_i}{k_b}
\]

\[
\frac{g_i - n_i}{n_i} = \exp \left( \frac{\alpha + \beta E_i}{k_b} \right)
\]

\[
n_i = \frac{g_i}{\exp \left( \frac{\alpha + \beta E_i}{k_b} \right) - 1}
\]

Substituting this back into \( f(n_i) \) yields

\[
k_b \ln W = \alpha N + \beta E
\]

Taking derivatives w.r.t. \( E \) gives

\[
dE = k_b \frac{\partial \ln W}{\partial E} = \frac{\alpha dN}{\beta}
\]

Recognizing that \( S = k_b \ln W \) and using the 2nd law of thermodynamics

\[
dE = T dS + \mu dN
\]

gives \( \beta = \frac{1}{T} \) and \( \alpha = -\frac{\mu}{T} \)
so that
\[ \langle N \rangle = \frac{ni}{\xi} = \frac{1}{\exp\left(\frac{E - \mu}{k_i T}\right) - 1} \]

which is the Fermi-Dirac function, with \( \mu \) being the chemical potential which we often substitute with the quasi-Fermi level \( E_F \) to get the usual form
\[ f(E) = \frac{1}{\exp\left(\frac{E - E_F}{k_i T}\right) - 1} \]

If we remove the limitation that only one particle is allowed per state (or box) then we have to count how many different ways we can arrange \( n_i + g_i - 1 \) things
\[ w(n_i, g_i) = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \]

Again we maximize entropy subject to the constraints that \( N = \sum n_i \) and \( E = \sum n_i E_i \) by forming a function \( f(n_i) \) and setting \( \frac{df}{dn_i} = 0 \)
\[ f(n_i) = k_B \ln \left[ \frac{(n_i + g_i - 1)!}{(g_i - 1)! n_i!} \right] + \alpha (N - \sum n_i) + \beta (E - \sum E_i) \]
\[ \eta_B = k_B \sum_i \left[ \ln \left( n_i + g_i - 1 \right) \right] - \ln \left( g_i - 1 \right) \]

\[ - \ln \left( n_i \right) + \alpha (N - E, n_i) + \beta (E - \Sigma, n_i) \]

\[ = k_B \sum_i \left( n_i + g_i - 1 \right) \ln \left( n_i + g_i - 1 \right) \]

\[ - \left( g_i - 1 \right) \ln \left( g_i - 1 \right) - n_i \ln \left( n_i \right) + \alpha (N - E, n_i) + \beta (E - \Sigma, n_i) \]

\[ + E \ln \left( n_i \right) \]

\[ \frac{\partial \eta}{\partial n_i} = k_B \left[ \frac{n_i + g_i - 1}{n_i + g_i - 1} + \ln \left( n_i + g_i - 1 \right) \right] \]

\[ - 1 - \ln \left( n_i \right) \]

\[ \alpha + \beta E_i = 0 \]

\[ \ln \left( \frac{n_i + g_i - 1}{n_i} \right) = \frac{\alpha + \beta E_i}{k_B} \]

\[ n_i = \frac{g_i - 1}{\exp \left[ \frac{\alpha + \beta E_i}{k_B} \right] - 1} \]

Again, substituting in \( \alpha = -\frac{E}{k_B T} \), \( \beta = \frac{E}{k_B T} \) and using \( g_i \gg 1 \), we get the Bose-Einstein distribution:

\[ N(E, T) = \frac{1}{\exp \left[ \frac{E - \mu}{k_B T} \right] - 1} \]
If we remove the requirement that particle number is conserved, then we get rid of the \( z(N=\text{even}) \) multiplier. In that case, we end up with the Planck distribution

\[
N(E,T) = \frac{1}{\beta} \exp \left( \frac{E}{\beta \kappa T} \right) - 1
\]

which is appropriate for phonons and photons as they do not have a Fermi level or chemical potential due to their number not being conserved.