Note Title 4/1/2010

Renormalization 4

Logic

Unknown H.E. $\rightarrow$ ~ local $\rightarrow$ term in $\mathcal{L}$ $\rightarrow$ renorm of λ, e, m ..

↑ Infinities ↑ measure

Regularization

- make loop integrals finite
- calculate

$\rightarrow$ undo regularization at end

Methods

1) Pauli Villars – cutoff

$$\frac{1}{p^2-m^2} - \frac{1}{p^2-\Lambda^2} = \frac{1}{p^2-m^2}\,\frac{m^2-\Lambda^2}{p^2-\Lambda^2}$$

small $\quad \Lambda \to \infty$

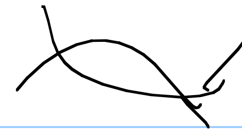
- sometimes destroys symmetries
  - gauge inv $\quad (m_\gamma = 0 \;\; \text{vs} \;\; m = \Lambda)$
  - chiral symm.

2) Dim reg
- 4 Dim $\to$ d dims $\quad d < 4 \qquad \int d^d l \, \frac{1}{(l^2-m^2)^2} \to$ finite

at end $d \to 4$

- preserves symmetries
- $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \quad$ tough

Integral in Pauli Villars



$$I(0) = \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l^2 - m^2)^2} \longrightarrow - \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - m^2)^2} \frac{\Lambda^4}{(l^2 - \Lambda^2)^2}$$

Feyman parameterize

$$I(0) = -\Lambda^4 \int \int \frac{dz_1\, dz_2\; z_1 z_2\; \delta(1 - z_1 - z_2)}{[l^2 - z_1 \Lambda^2 - z_2 m^2 + i\varepsilon]^4} \frac{d^4 l}{(2\pi)^4}$$

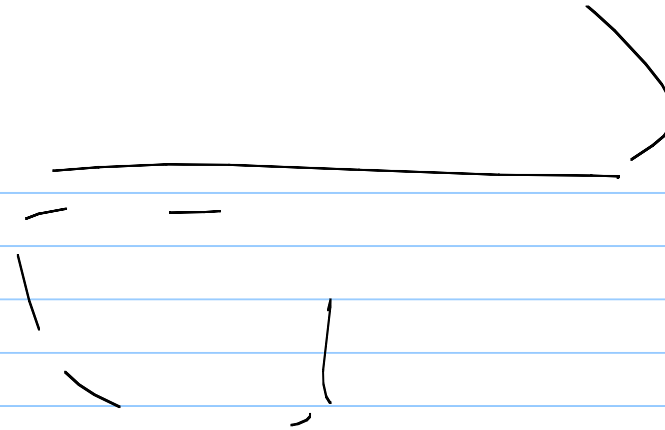
$$\frac{1}{a\, b\, c \cdots a_n} = (n-1)! \int \frac{dz_1\, dz_2 \cdots \quad \delta(1 - z_1 - z_2 - \cdots)}{[a z_1 + b z_2 \cdots a_n z_n]^n}$$

$$\frac{1}{a^2 b c \cdots} = n! \int \frac{dz_1\, z_1\; dz_2 \cdots \quad \delta(\quad)}{[a z_1 \cdots\quad]^{n+1}}$$

Wick rotate

$$l^2 = l_0^2 - \vec{l}^{\,2}$$

$$l_0 = i l_4 \qquad = \overbrace{-(l_4^2 + \vec{l}^{\,2})}$$



$$\Pi(0) = -i\Lambda^4 \int dz\, z(1-z) \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{\left[l_E^2 + \Lambda^2 z + m^2(1-z)\right]^4}$$

4D spherical coord

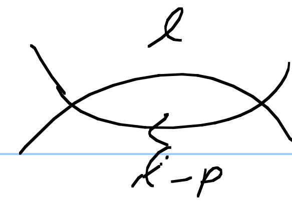
$$\int d^4 l_E = \int l^3\, dl\, \underbrace{d\Omega_{4D}}_{2\pi^2}$$

$$\Pi(0) = -i\Lambda^4 \frac{2\pi^2}{(2\pi)^4} \int dz\, z(1-z) \int l^3\, dl\, \frac{1}{\left[l^2 + \Lambda^2 z + m^2(1-z)\right]^4}$$

$$= \frac{-i}{96\pi^2}\left[\ln \Lambda^2/m^2 - 2\right]$$ as $\Lambda \to \infty$ drop $\frac{1}{\Lambda^2}$

Dimensional regularization

$4 \to d$



$$I(p) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 - m^2 + i\varepsilon} \frac{1}{(l-p)^2 - m^2 + i\varepsilon}$$

$$= \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2 + i\varepsilon} \frac{1}{(l-p)^2 - m^2 + i\varepsilon}$$

arbitrary, dimensions of mass , keeps dimensions same

Feynman parameterize

$$I(p) = \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \int_0^1 dz \frac{1}{\Big[\underbrace{\left[(l-p)^2 - m^2\right] z + \left[(l^2 - m^2)(1-z)\right] + i\varepsilon}_{\left[l^2 - 2 l\cdot p\, z + p^2 z - m^2 + i\varepsilon\right]}\Big]^2}$$

Shift variable

$$k = (\ell - pz) \implies k^2 = \ell^2 - 2\ell \cdot pz + p^2 z^2$$

$$[\ \ ] \to [k^2 + p^2 z(1-z) - m^2 + i\varepsilon] = [k^2 - a^2 + i\varepsilon]$$

Wick rotate $k_0 \to i k_4$

$$I(p) = i\,\mu^{4-d} \int dx \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[k^2 + a^2 - i\varepsilon]^2}$$

$$k^{d-1}\,dk\ \ d\Omega_d$$

Angles

$$\int d\Omega_d = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta_1 \, d\theta_1 \int_0^{\pi} \sin^2\theta_2 \, d\theta_2 \cdots \int \sin^{d-2}\theta_{d-2} \, d\theta_{d-2}$$

$$= 2 \frac{\Gamma(1/2)}{\Gamma(1)}\pi^{1/2} \frac{\Gamma(1)}{\Gamma(3/2)}\pi^{1/2} \frac{\Gamma(3/2)}{\Gamma(4/2)}\pi^{1/2} \cdots \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d/2)}\pi^{1/2}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad \leftarrow \text{use for any } d$$

Gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt \qquad (\text{Euler integral})$$

$\Gamma(1) = 1$; $\Gamma(-n) = \infty$

$\Gamma(z) = (z-1)\,\Gamma(z-1) \leftarrow$; $\Gamma(0) = \infty$

$\Gamma(n) = (n-1)!$; $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma$ $\leftarrow$ Euler #
$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} - \ln n\right]$

$\uparrow \varepsilon \to 0$ $= 0.577$

$\Gamma(1/2) = \sqrt{\pi}$

Checks:

$d=2$, $\Gamma(1)=1 \Rightarrow \int d\Omega_2 = 2\pi$ ✓

$d=3$ $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} = \int d\Omega_3 = \frac{2\pi^{3/2}}{\frac{1}{2}\sqrt{\pi}} = 4\pi$

$d=4$ $\Gamma(2)=1 \Rightarrow \int d\Omega_4 = 2\pi^2$

$$I(p) = i\mu^{4-d}\int dx \times \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(2\pi)^d}\int_0^\infty dk\, k^{d-1}\frac{1}{[k^2+a^2-i\varepsilon]^2}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^\infty dt\, \frac{t^{2x-1}}{(t^2+1)^{x+y}}$$, $x = d/2$, $x+y=2$

$y = 2 - d/2 = \frac{4-d}{2}$

$$I(p) = i \frac{\mu^{4-d}}{16\pi^2 (4\pi)^{\frac{d-4}{2}}} \Gamma\left(\frac{4-d}{2}\right) \int dx \frac{1}{a^{\frac{4-d}{2}}}$$

$a = m^2 - p^2 x(1-x)$

$\Gamma(0) = \infty$

$$\varepsilon = \frac{4-d}{2}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma$$

$$a^{\varepsilon} = e^{\varepsilon \ln a} = 1 + \varepsilon \ln a \qquad \left(\frac{1}{\varepsilon} a^{\varepsilon} = \frac{1}{\varepsilon} + \ln a \right)$$

$$I(p) = \frac{i}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma\right] \int dx \left[1 + \varepsilon \ln \mu^2 - \varepsilon \ln a - \varepsilon \ln 4\pi\right]$$

$$= \frac{i}{16\pi^2}\left\{\left[\frac{1}{\varepsilon} - \gamma - \ln 4\pi + \ln \mu^2/m^2\right] + \int dx \ln\left[\frac{m^2}{m^2 - p^2 x(1-x) - i\varepsilon}\right]\right\}$$

$$I(0)$$

$$\underbrace{\Delta I(p) = I(p) - I(0)}_{\text{same as shown before}}$$

$$\ln \Lambda^2 \to \frac{1}{\varepsilon}$$

when $\frac{1}{\varepsilon}$ disappears $\ln \mu^2$ + $\ln 4\pi$ disappear also

$\Rightarrow$ physics independent of μ