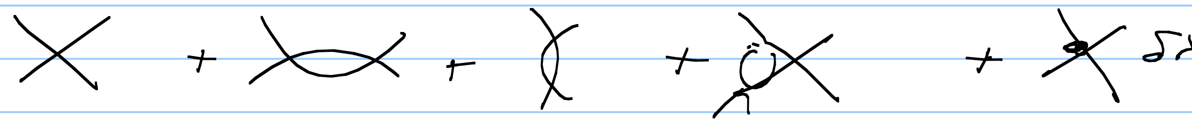
Renormalization 2

Scattering 

$$-i\mathcal{M} = -6i\lambda_{Rc} + \frac{1}{2}(-6i\lambda)^2\Big[\underbrace{I(s)}_{(p_1+p_2)^2} + \underbrace{I(t)}_{(p_1-p_3)^2} + \underbrace{I(u)}_{(p_1-p_4)^2} - 3I(0)\Big]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{128\pi^2 s}\,|\mathcal{M}|^2$$

$$\Delta I = \Big[I(p) - I(0)\Big] = \int \frac{dk^4}{(2\pi)^4}\Big[\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(k-p)^2-m^2+i\varepsilon} - \frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{k^2-m^2+i\varepsilon}\Big]$$

$$= -\int \frac{d^4k}{(2\pi)^4}\,\frac{(2k\cdot p - p^2)}{\left[k^2-m^2+i\varepsilon\right]^2\underline{\left((k-p)^2-m^2+i\varepsilon\right)}}$$

Feynman parameterization

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 \frac{dz_1\, dz_2 \cdots dz_n\, \delta(1-z_1-z_2-z \cdots z_n)}{(a_1 z_1 + a_2 z_2 \cdots + a_n z_n)^n}$$

Diff w.r.t. a_1

$$\frac{1}{a_1^2 a_2 \cdots a_n} = n! \int_0^1 \frac{z_1\, dz_1\, dz_2 \cdots dz_n\, \delta(1-z_1 \cdots z_n)}{(\quad)^{n+1}}$$

For us

$$\frac{1}{a_1^2 a_2} = \int_0^1 \frac{z_1\, dz_1\, dz_2\, \delta(1-z_1-z_2)}{(a_1 z_1 + a_2 z_2)^3} = \int_0^1 \frac{(1-z_2)\, dz_2}{(a_1(1-z_2)+a_2 z_2)^3}$$

$$\frac{1}{(k^2-m^2+i\varepsilon)^2((k-p)^2-m^2+i\varepsilon)} = \int_0^1 \frac{dz\,(1-z)}{\underbrace{((k^2-m^2)(1-z)+z((k-p)^2-m^2+i\varepsilon))}_{D}{}^3}$$

$$D = k^2 - \underline{2k\cdot p\, z} + (z p^2 - m^2) + i\varepsilon$$

Complete the square:

$$l = k - zp \quad , \quad l^2 = k^2 - 2k\cdot p\, z + z^2 p^2$$

$$D = l^2 + z(1-z)p^2 - m^2 + i\varepsilon \equiv l^2 - \underset{\uparrow\, m^2 - z(1-z)p^2}{a^2} + i\varepsilon$$

Use $\int d^4 l = \int d^4 k$

$$2k\cdot p - p^2 = 2l\cdot p - (1-2z)p^2$$

$$\Delta I = -\int_0^1 dz\,(1-z)\int \frac{d^4 l}{(2\pi)^4}\,\frac{\left[\cancelto{0}{2l\cdot p} - (1-2z)p^2\right]}{\left[l^2 - a^2 + i\varepsilon\right]^3}$$

$$= p^2\int_0^1 dz\,(1-z)(1-2z)\int \frac{d^4 l}{(2\pi)^4}\,\frac{1}{\left[l^2 - a^2 + i\varepsilon\right]^3}$$

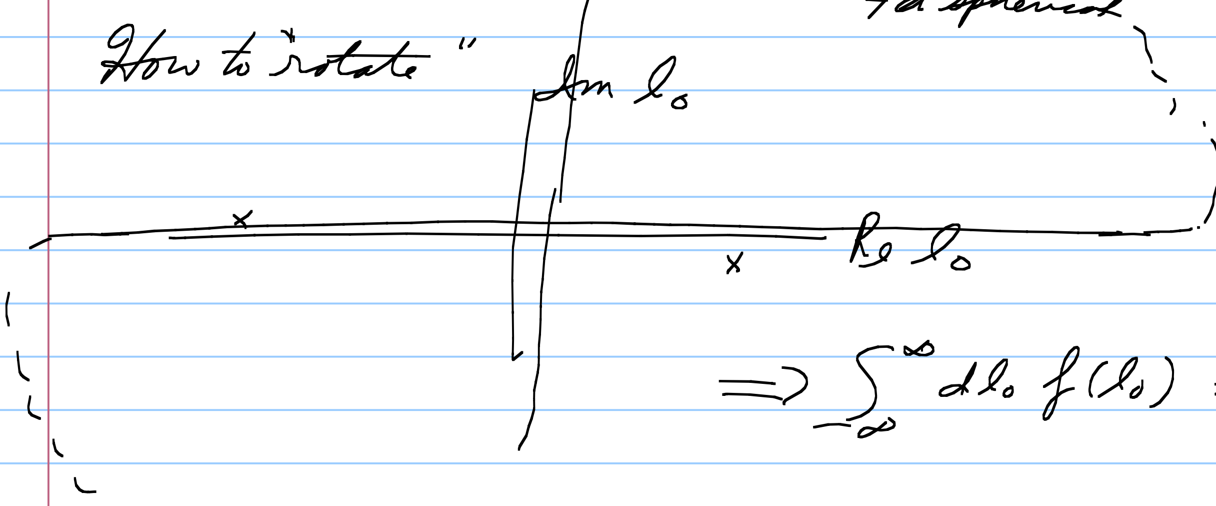
$$J_3 = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - a^2 + i\varepsilon]^3}$$

$$\underline{\underline{l_0^2}} - \vec{l}^{\,2} - a^2 + i\varepsilon \quad \Leftarrow \quad l_0 = \sqrt{\vec{l}^{\,2}+m^2} - i\varepsilon, \quad -\sqrt{\vec{l}^{\,2}+m^2} + i\varepsilon$$

"Wick rotate"

$$l_0 = i l_4 \qquad \Rightarrow \Big[\underbrace{-l_4^2 - \vec{l}^{\,2}}_{\text{4d spherical}} - a^2 + i\varepsilon\Big]$$

How to "rotate"



No poles

$$\Rightarrow \oint dl_0\, f(l_0) = 0$$

$$l_0 = i l_4$$

$$\Rightarrow \int_{-\infty}^{\infty} dl_0\, f(l_0) = i \int_{-\infty}^{\infty} dl_4\, f(i l_4)$$

$$l_E = (l_1, l_2, l_3, l_4)$$

$$J_3 = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 - a^2 + i\varepsilon} = -i \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{\left[l_E^2 + a^2 - i\varepsilon\right]^3}$$ ✓

<u>4D integral</u>

$$\int d^4 l_E = \int_0^\infty l_E^3 \, dl_E \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\pi} \sin^2\chi \, d\chi \to 2\pi^2 \int_0^\infty l_E^3 \, dl_E$$

$$J = \frac{-i \, 2\pi^2}{(2\pi)^4} \int_0^\infty \frac{l^3 \, dl}{(l^2 + a^2 - i\varepsilon)^3} = \frac{-i}{32\pi^2 (a^2 - i\varepsilon)}$$

Overall

$$\Delta I(p) = \frac{-i}{32\pi^2} p^2 \int_0^1 dz \frac{(1-z)(1-2z)}{\left[m^2 - z(1-z)p^2 - i\varepsilon\right]}$$ ✓

Do last integral over Feynman parameters

— use Gradstein & Ryzhik

$$\Delta I(p^2) = \frac{-i}{32\pi^2}\left\{ 2 + \left(\frac{4m^2 - p^2}{|p^2|}\right)^{1/2} \ln\left(\frac{\sqrt{4m^2-p^2} - |p|}{\sqrt{4m^2-p^2} + |p|}\right) \right\} \qquad p^2 < 0$$

$$= \frac{-i}{32\pi^2}\left\{ 2 - 2\left(\frac{4m^2-p^2}{p^2}\right)^{1/2} \tan^{-1}\left(\sqrt{\frac{p^2}{4m^2-p^2}}\right) \right\} \qquad 0 < p^2 < 4m^2$$

$$= \frac{-i}{32\pi^2}\left\{ 2 + \left(\frac{p^2 - 4m^2}{p^2}\right)^{1/2} \left[\ln\left(\frac{p - (p^2-4m^2)^{1/2}}{p + \sqrt{p^2-4m^2}}\right) + i\pi\right] \right\} \qquad p^2 > 4m^2$$

Wow!

$p^2 \Rightarrow$ $s = (p_1 + p_2)^2$, $u = (p_1 - p_4)^2$

$t = (p_1 - p_3)^2$