Note Title 4/8/2010

QED 1

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi + (D_\mu \phi)^*(D^\mu \phi) - V(\phi^*\phi)$$

$$D_\mu = \partial_\mu + i g A_\mu$$

Charge quantization

$q = ne$ (really $q = \frac{ne}{3}$)

— not predicted in QED

q_0 quantized $\Rightarrow$ q_R quantized !! "Ward Identities"

Gauge invariance - in matrix elements

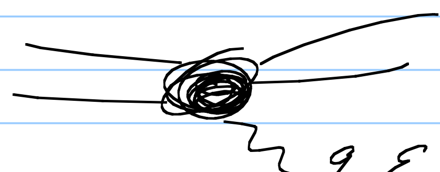
$$A_\mu \to A_\mu + \partial_\mu \Lambda(x)$$

$$A_\mu(x) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[a(q)\,\epsilon_\mu(q,\lambda)\,e^{-iq\cdot x} + h.c.\right]$$

$$\partial_\mu \Lambda(x) = \int \frac{d^4q}{(2\pi)}\,\Lambda(q)\,e^{-iq\cdot x}\,(-i)\,q_\mu$$

$$\Rightarrow \quad \epsilon_\mu(q,\lambda) \to \epsilon_\mu(q,\lambda) + \#\, q_\mu$$

Matrix element

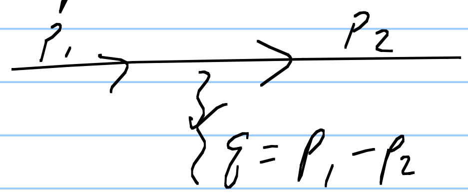


$q,\ \epsilon_\mu(q,\lambda)$

$$-i\mathcal{M} = -i\,\epsilon_\mu(q,\lambda)\,\mathcal{M}^\mu$$

$$\Rightarrow \quad q_\mu \mathcal{M}^\mu = 0$$

Example Scalar QED



$$= -ie\,(p_1 + p_2)^\mu \,\varepsilon_\mu$$

Then $q_\mu \mathcal{M}^\mu = q_\mu (-ie)(p_1 + p_2)^\mu = (-ie)(\overset{m^2}{p_1^2} - \overset{m}{p_2^2}) = 0$

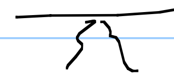
- Fermions $\mathcal{M}_\mu = -ie\, \bar{u}(p_2)\, \gamma^\mu\, u(p_1)$

$$q^\mu \mathcal{M}^\mu = -ie\, \bar{u}(p_2)\left[\overset{m}{\not{p}_1} - \overset{m}{\not{p}_2}\right] u(p_1) = 0 \checkmark$$

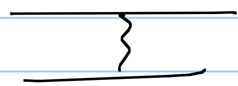
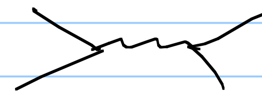
Solving QED

1) Read $\mathcal{L}$

 $= -ie\gamma_\mu$ or $-ie(p_1+p_2)^\mu$



2) Tree level processes

 + 

3) Renormalize at 1 loop

4) Predictions at 1 loop

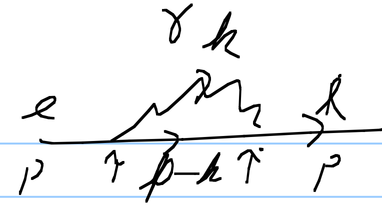
5) 2 loop ...

Mass + wavefunction for f

n   for γ

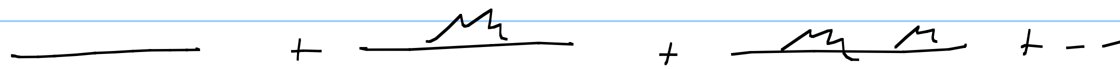
charge renorm e

Electron mass + wavefunction



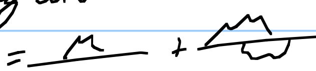
$= -i\Sigma(p)$

Summing



$$\frac{i}{\not p - m_0 + i\epsilon} \longrightarrow \frac{i}{\not p - m_0 - \Sigma(p) + i\epsilon}$$

Higher orders

$-i\Sigma =$ 

Feynman rules

$$-i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} \left(-ie\gamma_\mu \frac{i}{\not p - \not k - m + i\epsilon} -ie\gamma_\mu \right) \frac{-i}{k^2 + i\epsilon}$$

Expand $\leftarrow$ (Z_1, Z_2, Z_3 in QED)

$$\Sigma(p) = \delta m - (Z_2^{-1} - 1)(\not p - m) + \cdots$$

Then

$$\frac{i}{\not p - m_0 - \Sigma} = \frac{i Z_2}{\not p - m_{ren} + i\epsilon}$$

$$\delta m = \frac{3\alpha}{4\pi} \, m_0 \, \ln \Lambda^2 / m_0^2 \qquad \leftarrow \text{use Pauli-Villars}$$

$\uparrow \alpha / m_0$ $\qquad$ $\uparrow \frac{2}{4-d}$ in dim reg

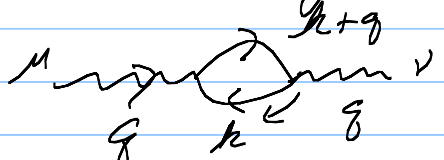
$\Rightarrow$ if $m_0 = 0$, $\delta m = 0$ (<u>chiral symmetry</u>) $\leftarrow$ protects $m = 0$

Z_2 in wavefunction renorm

$$\left(Z_2^{-1} - 1\right) = - \frac{\partial \Sigma}{\partial \cancel{p}} \qquad \checkmark$$

(Ward ID $Z_1 = Z_2$)

Photon Wavefunction Renorm — Vacuum Polarization



$$\equiv i\Pi_{\mu\nu}(q)$$

all charged particles

For fermions

closed fermion loops

$$i\Pi^{\mu\nu}(q) = -\int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[-ie\gamma_\mu \frac{i}{\not k + \not q - m} -ie\gamma_\nu \frac{i}{\not k - m} \right]$$

Gauge invariance

$$q_\mu \Pi^{\mu\nu} = 0$$

Check

$$q_\mu \, i\Pi^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\frac{1}{\not{k}-m} \, \not{q} \, \frac{1}{\not{k}+\not{q}-m} \, \gamma^\nu \right]$$

$$\not{q} = (\not{k}+\not{q}-m) - (\not{k}-m)$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\frac{1}{\not{k}-m} \gamma^\nu - \frac{1}{\not{k}+\not{q}-m} \gamma^\nu \right] \longrightarrow 0$$

$$\left. \begin{array}{l} \text{redefine } (k+q) = l \\ d^4k = d^4l \end{array} \right\} \quad q_\mu \Pi^{\mu\nu} = 0 \; \checkmark$$

<u>But</u>: With cutoff $\int^\Lambda d^4k \equiv \int^\Lambda d^4(k+q)$

With Pauli Vilars

$$\left(\frac{1}{\not{k}-m} - \frac{1}{\not{k}-\Lambda} \right) \not{q} \left(\frac{1}{\not{k}+\not{q}-m} - \frac{1}{\not{k}+\not{q}-\Lambda} \right) \longrightarrow \text{do not cancel} \qquad q_\mu \Pi^{\mu\nu} \neq 0$$

So end up $\qquad \Pi^{\mu\nu}(q^2=0) = \# \Lambda^2 g^{\mu\nu}$

$\nwarrow$ photon mass $\propto \Lambda^2$

$\Rightarrow$ Cutoffs & P.V. violate gauge invariance!

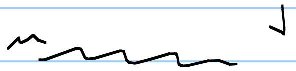
Old style – remove by hand

New style – use dim. reg.

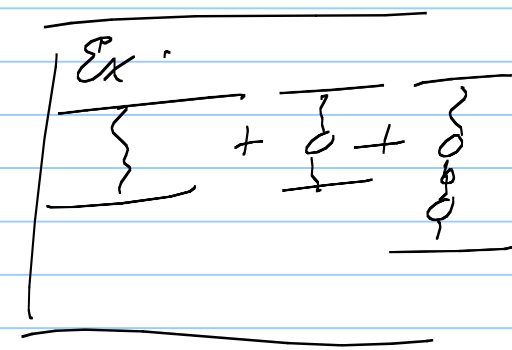
Structure

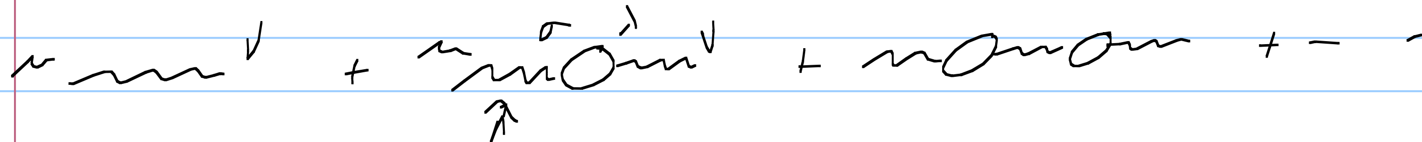
$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2) \qquad q_\mu \Pi^{\mu\nu} = 0 \ ✓$$

Propagator

 $= \dfrac{-i g^{\mu\nu}}{q^2 + i\varepsilon}$

Ex:



 $+ \dots$

$$\frac{-i g^{\mu\nu}}{q^2 + i\varepsilon} + \frac{-i g^{\mu\sigma}}{q^2 + i\varepsilon}\, i\left(q^2 g^{\sigma\lambda} - q^\lambda q^\sigma\right)\Pi(q^2)\, \frac{-i g^{\lambda\nu}}{q^2 + i\varepsilon}$$

$$= \frac{-i g^{\mu\nu}}{q^2}\left[1 + \Pi(q^2) + \Pi^2(q^2) + \dots\right] + \frac{q^\mu q^\nu \Pi(q^2)}{q^4}$$

disappear

$q^\mu \mathcal{M}^\mu = 0$

Sum:

$$-\frac{i\, g_{\mu\nu}}{q^2(1-\Pi(q^2))}$$

Define near pole

$$\Pi(q^2) = \underline{\underline{\Pi(0)}} + q^2\, \underline{\underline{\Pi'(q^2)}} + \cdots$$

$$\frac{-i\, g_{\mu\nu}}{q^2(1-\Pi(0))} = -i\,\frac{g_{\mu\nu}}{q^2}\, Z_3 \qquad Z_3 = \frac{1}{1-\Pi(0)} \quad \text{w.f. ren.}$$

$$A_\mu^0 = Z_3^{1/2} A_\mu^{ren}$$

Calculating Vac. Pol.

$$\mathrm{Tr}\left[\gamma_\mu \frac{1}{\not{k}+\not{q}-m}\gamma_\nu \frac{1}{\not{k}-m}\right] = \frac{1}{k^2-m^2}\frac{1}{(k+q)^2-m^2}\mathrm{Tr}\left[\gamma_\mu(\not{k}+\not{q}+m)\gamma_\nu(\not{k}+m)\right]$$

$$\frac{1}{\not{k}-m} * \frac{\not{k}+m}{\not{k}+m} = \frac{\not{k}+m}{(\not{k})^2-m^2} = \frac{\not{k}+m}{k^2-m^2}$$

$\not{k}\not{k} = k^2$

$\not{A}\not{A} = A^2$

Traces

$\mathrm{Tr}[\mathbb{1}] = 4$

$\mathrm{Tr}[\gamma^\mu] = 0$

$\mathrm{Tr}[\not{a}\not{b}] = \frac{1}{2}\mathrm{Tr}[\not{a}\not{b}+\not{b}\not{a}] = \frac{1}{2}\mathrm{Tr}[2g_{\mu\nu}a^\mu b^\nu] = 4a\cdot b$

$\mathrm{Tr}[\text{odd }\#] = 0$

$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[a\cdot b\, c\cdot d + a\cdot d\, b\cdot c - (a\cdot c)(b\cdot d)]$

$$\Pi^{\mu\nu} = i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+q)^2-m^2} 4\left[k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}\left(m^2 - k\cdot(k+q)\right)\right]$$

use dim reg

$$= \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \frac{e^2}{2\pi} \frac{\Gamma(\varepsilon/2)\,\mu^\varepsilon}{(4\pi)^{-\varepsilon/2}} \int_0^1 dx \frac{x(1-x)}{\left[m^2 - q^2 x(1-x)\right]^{\varepsilon/2}}$$

$\uparrow$ div. $\frac{1}{\varepsilon}$, $a^\varepsilon = e^{\varepsilon \ln a} = 1 + \varepsilon \ln a$

$\swarrow \varepsilon = 4 - d$

$$\Pi(q^2) = \frac{e^2}{6\pi^2} \left\{ \frac{1}{\varepsilon} + \frac{1}{2}\ln 4\pi - \frac{\gamma}{2} - 3\int dx\, x(1-x) \ln\left[\frac{m^2 - q^2 x(1-x)}{\mu^2}\right] \right\}$$

$$= \left\{ \quad '' \quad - \frac{1}{2}\ln m^2/\mu^2 + \frac{q^2}{10 m^2} \right\} \qquad q^2 \ll m^2$$

$$= \left\{ \quad '' \quad + \frac{5}{6} - \frac{1}{2}\ln\left(\frac{-q^2}{\mu^2}\right) \right\} \qquad |q|^2 \gg m^2 ,$$