Path Integrals 5

Note Title 5/4/2010

P.I. + Quantum Stat Mech

A) QM

Started with

$$D(x_f, t_f; x_i, t_i) = \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle$$

$$t \to -i\tau \quad \Rightarrow e^{-iHt} \to e^{-H\tau}$$

$\tau_i = 0$, $\tau_f = \beta$ with $\beta = \frac{1}{T}$ $(k_B = 1)$

End with $\langle x_f | e^{-\beta H} | x_i \rangle$

↑ coord space rep

$\hat{\rho} = e^{-\beta H}$ - density matrix

Partition function

$$Z = \mathrm{Tr}\, e^{-\beta H}$$
$$= \int d^3x \, \langle x | \, e^{-\beta H} \, | x \rangle$$

Identify $x_f = x_i$ or $x(\beta) = x(0)$ $\Leftarrow$ Periodic B.C.

Path Integral Rep

$$iS \longrightarrow -\int_0^\beta d\tau \left[\frac{1}{2} m \left(\frac{\partial x}{\partial \tau}\right)^2 + V(x) \right]$$

$$Z = \int \mathcal{D}x(\tau) \, e^{-\int_0^\beta d\tau \left[\frac{1}{2} m \left(\frac{\partial x}{\partial \tau}\right)^2 + V(x) \right]}$$

with $x(\beta) = x(0)$

Feynman Kac formula

as $T \to 0$, $\beta \to \infty$ (like $t \to \infty$) $\to$ Ground state only ✓

B) QFT

$x_i(t) \to \phi(x,t)$

$t \to -i\tau \qquad 0 \leq \tau \leq \beta$

$$\mathcal{L} + J\phi \to \Big[\underbrace{-\frac{1}{2}\Big(\frac{\partial\phi}{\partial\tau}\Big)^2 - (\vec{\nabla}\phi)^2 - V(\phi)}_{\text{looks like } -H} + J\phi\Big] = -\big[\mathcal{L}_E - J\phi\big]$$

Partition function

$$Z[J] = \int d\phi \, e^{-\int_0^\beta d^4x \, [\mathcal{L}_E - J\phi]}$$

Periodic BC $\quad \phi(x, \tau=\beta) = \phi(x, \tau=0)$

$\Rightarrow$ 4D Euclidean QFT on $0 < \tau < \beta$ (P.B.C) = Quantum Stat Mech in 3 spatial dim.

Note as $T \to 0 \quad Z[J] = Z[J]$!

Finite T Propagator

Pert Theory from $Z[J]$ $\Rightarrow$ Feynman rules related

– diff is propagator

need $\phi(x, \tau=\beta) = \phi(x, \tau=0)$

use Fourier sum $e^{-i\omega_n \tau}$ $\qquad \omega_n = \frac{2\pi n}{\beta}$

Want

$$\left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right) D_\beta(x, x') = \delta(\tau-\tau')\,\delta^3(x-x')$$

$$D_\beta(x) = \int \frac{d^3k}{(2\pi)^3}\, T \sum_n e^{+i\omega\tau} e^{i\vec{k}\cdot\vec{x}}\, g(\omega_n, k)$$

$$\Rightarrow \quad g(\omega_n, k) = \frac{1}{\omega_n^2 + k^2 + m^2}$$

$$\int \frac{d^4k_E}{(2\pi)^4} \frac{e^{ik_E \cdot x}}{k_0^2 + \vec{k}^2 + m^2} \longrightarrow T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_n \tau} e^{i\vec{k}\cdot\vec{x}}}{\omega_n^2 + \vec{k}^2 + m^2}$$

$$T\sum_n \sim \int \frac{dk_0}{2\pi} \quad \text{as } T \to 0$$

Feynman rules

$$\text{———} = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2}$$

$$\int d^4k \to 2\pi T \sum_n d^3k$$

Imag time formulation

(also real time formulation)

Effective Field Theory 101 – "integrating out" field

Idea

- some fields heavy, some light
- low energy light fields only
- heavy field in loops, propagators

$\Rightarrow$ do PI over heavy field

- residual effects in $\mathcal{L}_{eff}$.

Do field theory with light fields

Formulas

$$\phi \text{ light}, \quad \chi \text{ heavy}$$

$$\mathcal{L}(\phi, \chi) = \underbrace{\mathcal{L}_0(\phi)}_{\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)} + \underbrace{\mathcal{L}_0(\chi)} + \underbrace{\mathcal{L}_I(\phi, \chi)}_{\chi\phi^2,\ \chi^2\phi^2 + \cdots}$$

Full Theory

$$Z[J_\phi, J_\chi] = \int [d\phi][d\chi]\, e^{i\int d^4x \left[\mathcal{L}_0(\phi) + \mathcal{L}_0(\chi) + \mathcal{L}_I + J_\phi \phi + J_\chi \chi\right]}$$

Matrix elements of light field

$$G_\phi^{(n)}(1, \ldots) = \frac{(-i)^n}{Z[0]} \frac{\delta Z[J_\phi, 0]}{\delta J_\phi(x_1) \cdots \delta J_\phi(x_n)} \Bigg|_{J_\phi = 0}$$

Then

$$Z[J_\phi, 0] = \int [d\phi]\, e^{i\int d^4x [\mathcal{L}_0(\phi) + J_\phi \phi]} \underbrace{\int [d\chi]\, e^{i\int d^4y [\mathcal{L}_0(\chi) + \mathcal{L}_I(\phi, \chi)]}}_{e^{iW[\phi]}}$$

= integrating out

If we can write

$$e^{iW[\phi]} = e^{i\int d^4x\, \mathcal{L}_{extra}(\phi)}$$

$$Z[J_\phi] = \int [d\phi]\, e^{i\int d^4x [\mathcal{L}_{eff} + J\phi]} \leftarrow \text{Eff. Field Theory}$$

with $\mathcal{L}_{eff} = \mathcal{L}_0 + \mathcal{L}_{extra}$

If χ is heavy, can always write local $\mathcal{L}_{eff}$

uncertainty princ. $\Delta x \sim 1/\Delta E \sim 0$

if χ is light

cannot write $\mathcal{L}_{eff}$

Exampl Linear coupling

$$\mathcal{L}_I(\chi,\phi) = \chi \underbrace{F(\phi)}_{\phi^2}$$

$$\mathcal{L}_0(\chi) + \mathcal{L}_I(\chi,\phi) = \frac{1}{2}(\partial_\mu \chi)^2 - m^2\chi^2 + \chi F(\phi)$$

$$e^{iW[\phi]} = \int d\chi \, e^{i\int d^4x [\mathcal{L}_0 + \chi F(\phi)]}$$

$J \to F(\phi)$

$$= N \, e^{-\frac{i}{2}\int d^4x \, d^4y \, F(\phi(x)) D_F(x-y) F(\phi(y))}$$

To put in local form

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m_X^2 + i\epsilon} \underset{\uparrow \, m_X \text{ large}}{=} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left[\frac{-1}{m_X^2} - \frac{k^2}{m_X^4} + \dots \right]$$

$$= \left[-\frac{1}{m_X^2} + \frac{\Box}{m_X^4} + \dots \right] \underbrace{\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}}_{\delta^4(x-y)}$$

$$W[\phi] = -\frac{1}{2} \int d^4x \underbrace{\left[\frac{-1}{m_X^2} \left[F(\phi)\right]^2 + \frac{1}{m_X^4} F(\phi^2) \Box F(X^2) + \dots \right]}_{\text{local Lextra}}$$

Example (from Goldstone Bosons — "Calculating 4" Mar 11)

$$V(\phi_1, \phi_2) = -\frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2$$

Goldstone

Ground state $\phi_1 = v = \sqrt{\mu^2/\lambda}$, $m_1^2 = 2\mu^2$, $m_2^2 = 0$

Scattering amplitude — lots of cancellations

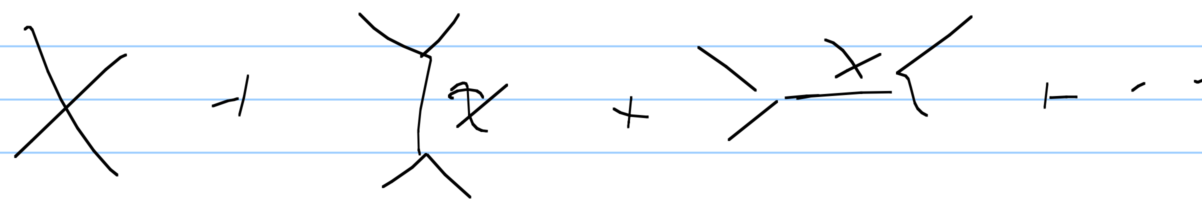
$$\phi_1 = \chi \quad , \quad \phi_2 = \phi$$

$$\ldots \quad \mathcal{L}_I(\phi, \chi) = \lambda v \chi \phi^2 + \ldots$$

$$W[\phi] = -\frac{1}{2}\int d^4x \left[-\frac{1}{2\mu^2}\left[\lambda v \phi^2\right]^2 + \frac{1}{(2\mu^2)^2}\lambda^2 v^2 \phi^2 \Box \phi^2 + \ldots \right]$$

$$= \int d^4x \; + \frac{\lambda}{4}\phi^4 - \frac{\lambda}{2m_\chi^2}\phi^2 \Box \phi^2$$

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}(\partial_\mu\phi)^2 \underbrace{- \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}\phi^4}_{\text{cancel!}} - \frac{\lambda}{2m_\chi^2}\underbrace{\phi^2 \Box \phi^2}_{\text{yield correct amp.}}$$



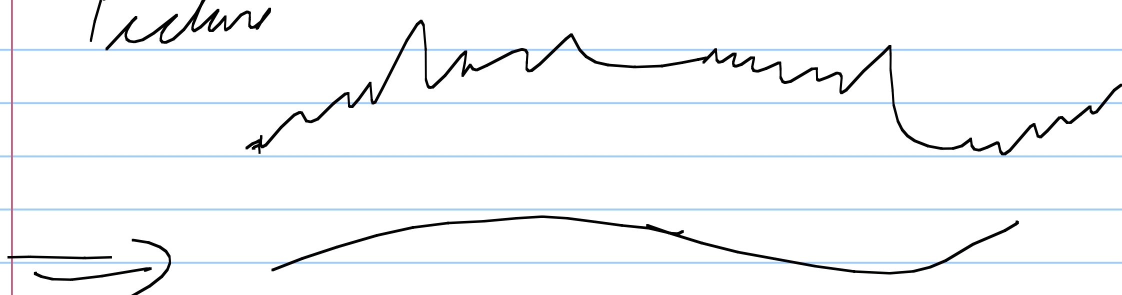
"Wilsonian" Renormalization group

- Ken Wilson
- based on "integrating out"

Idea

- even light field have components $E > \Lambda$
- integrate out all effects beyond Λ
- $\mathcal{L}_{eff}$ - applied below Λ

Picture



← field with fluct on all scales

← removed small scale

Formulas

$$Z[J] = \int \underbrace{[d\phi]}_{[d\phi]_{k<\Lambda}[d\phi]_{k>\Lambda}} e^{i\int d^4x[\mathcal{L} + J\phi]}$$

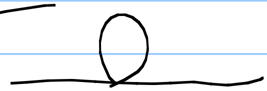
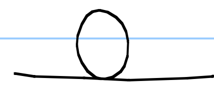
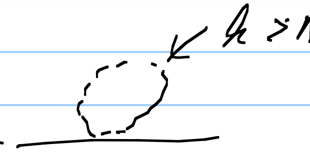
$$= \int [d\phi]_{k<\Lambda} e^{i\int d^4x[\mathcal{L}_{eff}(\Lambda) + J\phi]}$$

with $$e^{i\mathcal{L}_{eff}} = \int [d\phi]_{k>\Lambda} e^{i\int d^4 \mathcal{L}(\phi)}$$

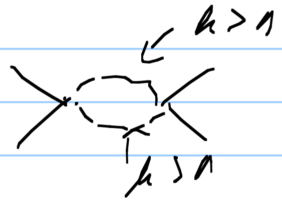
In Pert Theory

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} = \Big[\int_0^\Lambda \frac{d^4k}{(2\pi)^4} + \underbrace{\int_\Lambda^\infty \frac{d^4k}{(2\pi)^4}}_{\text{do explicitly}}\Big]\,\frac{i}{k^2 - m^2}$$

Get

 =  + 

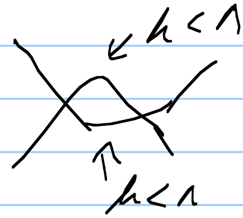
$$\Delta\mathcal{L} = \frac{\delta m^2(\Lambda)}{2}\,\phi^2$$



$$= \delta\lambda(\Lambda) + \underbrace{(p_1+p_2)^2}_{\text{like } \delta\mathcal{L} = \delta g_6\,\phi^2\Box\phi^2}\,\delta g_6(\Lambda) + \dots$$

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2(\Lambda)}{2}\phi^2 + \frac{\lambda(\Lambda)}{4}\phi^4 + g_6(\Lambda)\,\phi^2\Box\phi^2 + \dots$$

Do QFT below Λ (like Pauli-Villars)



$+ \cdots \quad \frac{1}{2}(-6i\lambda))\left[I_\Lambda(s) + I_\Lambda(t) + I_\Lambda(u)\right]$

Physical amplitude $\quad -i\mathcal{M}(s=t=u=0) = -6i\lambda_{ph}$

$$\lambda_{ph} = \lambda(\Lambda) - \frac{3}{16\pi^2}\lambda^2 \ln(\Lambda/m_{ph})$$

$\uparrow$ $\lambda_{ph} \approx \lambda(\Lambda)$ to this order

$$\lambda(\Lambda) = \lambda_{ph} + \frac{3}{16\pi^2}\lambda_p^2 \ln(\Lambda/m_{ph}) \quad \Longleftarrow \lambda(\mu) \text{ in HW}$$

$\Longrightarrow$ R.G. is the same