Path Integrals 2

Note Title 4/22/2010

$$Z[J] = \int dx_1 \dots dx_N \, e^{i\left[\frac{1}{2}\vec{x}\cdot A\cdot\vec{x} + J\cdot x\right]}$$

$$= \left[\frac{(2\pi i)^N}{\det A}\right]^{1/2} e^{-\frac{i}{2} J\cdot A^{-1}\cdot J} \qquad **$$

Warm up

$$\langle x_i x_j \rangle = \frac{\int [dx] \, e^{+\frac{i}{2} x\cdot A\cdot x} \, x_i x_j}{\int [dx] \, e^{\frac{i}{2} x\cdot A\cdot x}}$$

$$= -\frac{1}{Z[0]} \frac{\partial^2}{\partial J_i \partial J_j} Z[J] \Big|_{J=0}$$

$$= -\frac{1}{Z[0]} \frac{\partial}{\partial J_i} \left[-i A^{-1}_{i\ell} J_\ell \, e^{-\frac{i}{2} J\cdot A^{-1}\cdot J} \right]$$

$$= i A^{-1}_{ij} \quad \checkmark$$

$$\langle x_i x_j x_k x_l \rangle = \frac{1}{Z[0]} \frac{\delta^4}{\delta J_i \, \delta J_j \, \delta J_k \, \delta J_l} Z[J] \Big|_{J=0}$$

$$= i A^{-1}_{ij} \, i A^{-1}_{kl} + i A^{-1}_{ik} \, A^{-1}_{jl} + i A^{-1}_{il} \, i A^{-1}_{kj}$$

QFT

$$x_i \longrightarrow \phi(x)$$

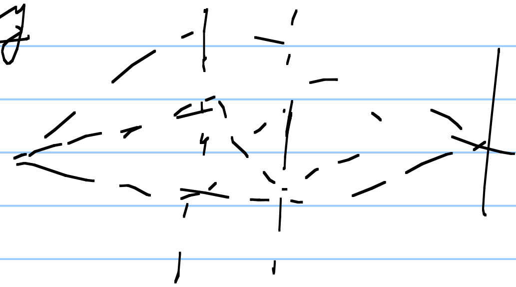
$$\int [dx] \longrightarrow \int [d\phi]$$

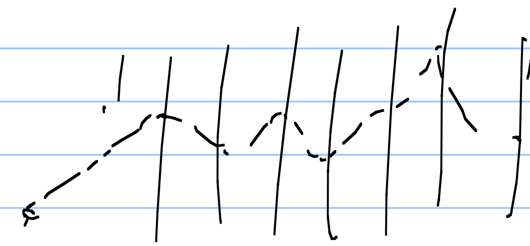
$$e^{\frac{i}{2} x \cdot A \cdot x} \longrightarrow e^{i \int d^4x \, \frac{1}{2} \phi (\Box + m^2) \phi}$$

$$\langle x_i \, x_j \rangle \longrightarrow \langle \phi(x) \, \phi(x') \rangle = i D_F(x - x') \qquad *$$

QM P.I

Story





Set up

$$\mathcal{M} = \langle \psi_f(t) | \, e^{-iH(t-t_0)} \, | \psi_i(t_0) \rangle$$

$$= \int dx_f \, dx_i \, \langle \psi_f(t) | x_f \rangle \underbrace{\langle x_f | \, e^{-iH(t-t_0)} \, | x_i \rangle} \langle x_i | \psi_i(t_0) \rangle$$

$$= \int dx_f \, dx_i \, \psi_f(x_f, t) \, D(x_f \, t \, ; x_i \, t_0) \, \psi_i(x_i, t_0)$$

$$D(x_f \, x_i ; t) = \langle x_f | \, e^{-iHt} \, | x_i \rangle$$

Derivation

- N time steps $\delta t = t/N$ (∞ of screens)
- insert a complete set $\int d^3x_i \, |x_i\rangle\langle x_i|$ (∞ of holes)

$$D(x_f, x_i, t) = \int d^3x_1 \cdots d^3x_N \, \langle x_f | e^{-iH\delta t} | x_N\rangle\langle x_N| e^{-iH\delta t} |x_{N-1}\rangle \cdots \langle x_1| e^{-iH\delta t} |x_i\rangle$$

$H = \frac{p^2}{2m}$ + evaluate

$$\langle x_{i+1} | e^{-i\frac{p^2}{2m}\delta t} | x_i\rangle = \int \frac{d^3p}{(2\pi)^3} \langle x_{i+1}| e^{-i\frac{p^2}{2m}\delta t} |p\rangle\langle p|x_i\rangle$$

$$e^{-i\vec{p}\cdot\vec{x}_{i+1}} \qquad e^{i\vec{p}\cdot\vec{x}_i}$$

$$= (2\pi i)^{\frac{1}{2}} \left(\frac{m}{\delta t}\right)^{\frac{3}{2}} e^{i\frac{m(\vec{x}_i - \vec{x}_{i+1})^2}{\delta t}}$$

Then

$$D(x_f, x_i, t) = \underbrace{\int d^3x_1 \cdots d^3x_N \left(\frac{2\pi i N}{\delta t}\right)^{\frac{3}{2}N}} e^{i\sum_i \frac{m}{2}\left(\frac{\vec{x}_{i+1} - \vec{x}_i}{\delta t}\right)^2 \delta t}$$

$$D(x_f, x_i, t) = N_{orm} \int [dx^3] \; e^{i\int dt \left[\underbrace{\frac{1}{2} m \vec{v}^2}_{L} - V\right]}$$

With a potential $e^{-iHt} = e^{-i\left(\frac{p^2}{2m} + V\right)t}$

Contents:

- Wavefunctions + energies $\quad H|n\rangle = E_n |n\rangle$

$$D(x_f, x_i, t) = \langle x_f | n\rangle \; e^{-i\underset{\uparrow E_n}{H} t} \; \langle n | x_i\rangle$$

$$= \sum_n \psi_n(x_f)\, \psi_n^*(x_i)\, e^{-iE_n t}$$

Matrix elements

$$\langle x_f, t_f | T\big(x(t_1)\; x(t_2) \dots\big) | x_i, t_i\rangle = N\int [dx] \; x(t_1)\; x(t_2) \dots e^{-iHt}$$

Project out ground state

$$H \to (1-i\varepsilon)H$$

$$(t_f - t_i) = \tau \longrightarrow \infty$$

$$\sum_n e^{-iE_n(1-i\varepsilon)\tau} \longrightarrow e^{-iE_0(1-i\varepsilon)\tau} + \text{exponentially small}$$

S_0

$$D(x_f, x_i, t\to\infty) = \int \underbrace{N[dx]}_{\mathcal{D}(x)} e^{+i\int_{-\infty}^{\infty} dt' L(1+i\varepsilon)} = \psi_0(x_f)\psi_0(x_i) e^{-iE_0\tau}$$

Matrix element

$$\underset{\uparrow \text{G.S.}}{\langle 0|} T(x(t_i) x(t_j)) |0\rangle = \lim_{\tau\to\infty} \frac{\langle x_f | T(x\ x) | x_i\rangle}{\langle x_f | x_i\rangle}$$

$$= \frac{\int \mathcal{D}(x)\; x(t_\mu)\, x(t_\nu)\, e^{i\int dt\, L}}{\int \mathcal{D}(x)\; e^{i\int dt\, L}}$$

Functional Differentiation

Recall $x_j \, e^{\sum_i J_i x_i} = \frac{\delta}{\delta J_j} e^{\sum_i J_i x_i}$

Continuum generalization

$$x(t_j)\, e^{\int dt\, J(t) x(t)} = \frac{\delta}{\delta J(t_j)} e^{\int dt\, J(t) x(t)}$$

Formal def:

$$\frac{\delta J(t')}{\delta J(t)} \equiv \delta(t-t')$$

$$\Rightarrow \frac{\delta}{\delta J(t)} \int dt'\, J(t')\, f(t') = \int dt'\, \delta(t-t')\, f(t') = f(t)$$

Generating functional

$$Z[J] \equiv \lim_{t_f - t_i \to \infty} \int \mathcal{D}(x)\, e^{i\int_{t_i}^{t_f} dt\left[L(x,\dot{x}) + J(t)\,x(t)\right]}$$

Take

$$(-i)^2 \frac{\delta^2 Z[J]}{\delta J(t_f)\,\delta J(t_i)} = \int \mathcal{D}(x)\; x(t_f)\,x(t_i)\; e^{i\int dt\,[\quad]}$$

Matrix elements

$$\langle 0|T\,\underbrace{x(t_f)\,x(t_i)}_{\substack{t_f \to t_i \\ \text{in examples}}}|0\rangle = \frac{(-i)^2}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(t_f)\,\delta J(t_i)}\Bigg|_{J=0}$$

Harmonic Oscillator (1D)

$- \omega^2 \to \underline{\underline{\omega^2 - i\varepsilon}}$ like $(1-i\varepsilon)H$

$$S = \int dt \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + J(t) x(t) \right]$$

$$= \int dt \left[-\frac{1}{2} m x \left[\frac{d^2}{dt^2} + \omega^2 \right] x + J x \right]$$

Previously

$x \circ A \cdot x + J \cdot x$

$\to J A^{-1} J$

Complete the square

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) D(t-t') = -\delta(t-t')$$

$$\tilde{x}(t) = x(t) + \frac{1}{m} \int dt' \, D(t-t') J(t')$$

such that

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) \tilde{x} = (\quad) x - \frac{1}{m} J(t)$$

S_0

$$S = \int dt \underbrace{\left[-\frac{1}{2} m \tilde{x} \left(\frac{d^2}{dt} + w^2 \right) \tilde{x} \right]}_{S[\tilde{x}, J=0]} - \frac{1}{2m} \int dt\, dt' J(t) D(t-t') J(t')$$

P.I.

$$Z[J] = \int \mathcal{D}(x)\, e^{iS[x,J]} = \int \underbrace{\mathcal{D}(x)}_{\mathcal{D}(\tilde{x})} e^{iS[\tilde{x}, J=0]} e^{-\frac{i}{2m} \iint J D J}$$

$$= Z[0]\, e^{-\frac{i}{2m} \int dt\, dt' J(t) D(t-t') J(t')}$$

Calculate

$$\langle 0 | T (x(t_1)\, x(t_2)) | 0 \rangle = \frac{(-i)^2}{Z[0]} \frac{\delta^2}{\delta J(x_1)\, \delta J(x_2)} Z[J] \Big|_{J=0}$$

$$= (-i)^2 \frac{-i}{m} D(t_2 - t_1)$$

$$\langle 0|x^2|0\rangle = \lim_{t_2 \to t_1} \frac{i}{m} D(t_2 - t_1)$$

To evaluate

$$D(t-t') = \int \frac{dE}{2\pi} \frac{1}{E^2 - w^2 + i\epsilon} e^{-iE(t-t')}$$

$$\left(\frac{d^2}{dt^2} + w^2\right) D = \int \frac{dE}{2\pi} \frac{-E^2 + w^2}{E^2 - w^2 + i\epsilon} e^{-iE(t-t')} = -\delta(t-t')$$

$i\epsilon$ for convergence

– do by contour integral

$$D(t-t') = \frac{-i}{2w} e^{-iw|t'-t|}$$

Then $\langle 0|x^2|0\rangle = \frac{1}{2mw}$