Introducing Fields 3

Note Title 2/4/2010

Nonrelativistic Field

ψ, ψ^* independent

$$\mathcal{L} = \frac{i}{2}\left(\psi^*(\overrightarrow{\partial_0} - \overleftarrow{\partial_0})\psi - \frac{1}{2m}(\vec{\nabla}\psi^*)(\vec{\nabla}\psi)\right) \begin{Bmatrix} -\psi^* V(x)\psi \\ -u_0(\psi^*\psi)(\psi^*\psi) \end{Bmatrix}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_0\psi^*} = \pi^* = -\frac{i}{2}\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\vec{\nabla}\psi^*} = -\frac{1}{2m}\vec{\nabla}\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i}{2}\partial_0\psi$$

$$\partial_0\left(\frac{\partial\mathcal{L}}{\partial\partial_0\psi^*}\right) + \vec{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial\vec{\nabla}\psi^*}\right) - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0$$

$$-\frac{i}{2}\partial_0\psi - \frac{1}{2m}\nabla^2\psi - \frac{i}{2}\partial_0\psi = 0 \quad \text{or} \quad i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m}\psi \quad \checkmark$$

Hamiltonian

$$\frac{i}{2}\psi^*\partial_0\psi \;---$$

$$\mathcal{H} = \underset{+i\frac{\psi^*}{2}}{\Pi}\,\partial_0\psi + \underset{-\frac{i}{2}\psi}{\Pi^*}\,\partial_0\psi^* - \mathcal{L} \qquad = + \frac{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}{2m}$$

sent by parts

$$H = \int d^3x\; \psi^* \frac{-\nabla^2}{2m}\psi \left\{ \begin{array}{l} + \psi^* V \psi \\ + u_0(\bar{\psi}\psi)^2 \end{array} \right.$$

Alternate Derivation "Non-rel reduction"

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \qquad \longleftarrow \checkmark$$

Let $\phi = \frac{1}{\sqrt{2m}} e^{-imt} \psi(x,t)$ $\qquad e^{-iEt} \sim e^{-imt + \dots}$

$$\partial_0 \phi = \frac{e^{-imt}}{\sqrt{2m}} \left(-im + \partial_0 \right) \psi$$

$$\mathcal{L} = \frac{1}{2m} \left[(+im + \partial_0)\psi^* \, (-im + \partial_0)\psi - \vec{\nabla}\psi^* \vec{\nabla}\psi - m^2 \psi^* \psi \right]$$

$$= \underbrace{\frac{i}{2} \psi^* \left(\partial_0 - \overleftarrow{\partial_0} \right) \psi - \frac{1}{2m} \vec{\nabla}\psi^* \nabla \psi}_{\mathcal{L}_{sch.}} + \frac{1}{2m} (\partial_0 \psi)^* (\partial_0 \psi)$$

$\partial_0 \psi \sim \frac{\nabla^2}{2m} \psi$

$\uparrow \frac{p^2}{2m}$

$\uparrow p^4/m^2$

$\longleftarrow$ small $\Rightarrow$ drop. $\checkmark$

Quantization

$$\psi(x) = \sum_n e^{-iE_n t}\,\psi_n(x)\, a_n = \int \frac{d^3p}{(2\pi)} e^{-i(Et - \vec{p}\cdot\vec{x})} a(p)$$

$$H = \int d^3x\, \psi^* \left[\frac{-\nabla^2}{2m} \{+V\}\right] \psi = \sum_n \psi_n^* [\quad] \psi_n\, a_n^\dagger a_n$$

$$= \sum_n E_n a_n^\dagger a_n$$

"Second quantization"
(no antiparticles)

After the fact

$$[\psi(x), \psi^*(x')]_{E.T} = \sum_n \psi_n^*(x')\psi_n(x) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

$$= \delta^3(x - x')$$

Non-rel Fermions

Creation op $b^+(p,s)$ ← spin

Pauli $\begin{cases} b^+(p,s)\, b^+(p,s)|0\rangle = 0 \\ b^+(p,s)\, b^+(p',s')|0\rangle = -b^+(p',s')\, b^+(p,s)|0\rangle \end{cases}$

$$\{ b^+(p,s), b^+(p',s')\} = 0 \qquad \{A, B\} = AB + BA$$

$$\{ b(p,s), b^+(p',s')\} = \delta_{pp'}\delta_{ss'}$$

$$\psi(x,t) = \sum_{p,s} \psi_{p,s}(x)\, b(p,s) \implies H = \sum_{p,s} E_{p,s}\, b^+(p,s)\, b(p,s)$$

$$\{ \psi(x,t), \psi^*(x',t)\}_{ET} = \delta^3(\vec{x}-\vec{x}')$$

Photons:

$A^\mu = (\phi, \vec{A})$ $\Leftarrow$ basic

$$\left.\begin{aligned}\vec{E} &= -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}\right\}\text{ derived}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{0i} = -E^i$$

$$F^{ij} = -\varepsilon^{ijk} B^k$$

$$\partial^\mu F_{\mu\nu} = 0 = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

Lorentz gauge $\partial_\mu A^\mu = 0 = \frac{\partial}{\partial t}\phi + \vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \Box A^\nu = 0$

Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right)$$

(Units here Heaviside Lorentz)

$$H_{HL} = \frac{1}{2}\left(E^2 + B^2\right)$$

$$H_G = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

Eq of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \underbrace{\partial_\mu A_\nu}} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad \Rightarrow \quad \partial^\mu F_{\mu\nu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\frac{4}{4} F_{\mu\nu}$$

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$$

Hamiltonian

$$\pi_\nu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\nu} = -F_{0\nu} = \begin{cases} = 0 & \nu = 0 \\ \underline{E}_i & \nu = i \end{cases}$$

$$A^0 = \phi$$

$$\mathcal{H} = \Pi_\nu \underline{\partial^0 A^\nu} - \mathcal{L} = -E_i(\partial_0 A_i - \vec{\nabla}\phi + \vec{\nabla}\phi) - \frac{1}{2}(E^2 - B^2)$$

$$= \underline{\underline{\frac{1}{2}(\vec{E}^2 + \vec{B})}} \checkmark \quad + \underbrace{\vec{E}_i \cdot \vec{\nabla}\phi}_{\text{int by parts}} \quad \phi(\underbrace{-\vec{\nabla}\cdot\vec{E}}_{\rho}) = \underbrace{\rho\phi}_{\text{energy}\checkmark} = 0$$

Quantization - subtle (right in P.I.)

- results nice

$\Box A_\mu = 0 \qquad , \qquad \partial_\mu A^\mu = 0$

Solutions $A_\mu = \underbrace{\varepsilon_\mu(p,\lambda)}_{\text{2 polarizations transverse}} e^{-i\vec{p}\cdot\vec{x}}$ $\qquad p^2 = 0 = E^2 - \vec{p}^{\,2}$

$p_\mu \cdot \varepsilon^\mu(p,\lambda) = 0$

$$A_\mu(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[\varepsilon_\mu(p,\lambda) e^{-ip\cdot x} \underline{\underline{a(p,\lambda)}} + \varepsilon^*_\mu(p,\lambda) e^{+ip\cdot x} a^\dagger(p,\lambda) \right]$$

$$H = \int d^3x \, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \, \omega_p^{\swarrow |\vec{p}|} \, a^\dagger(p,\lambda) a(p,\lambda) \qquad + E_0^{\swarrow 2 E_0 \text{scalar}}$$

Poynting vector

$$\vec{S} = (\vec{E} \times \vec{B}) \qquad\qquad \frac{\partial \mathcal{H}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

$$\underline{\bar{P}} = \int d^3x \, (\vec{E} \times \vec{B}) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \, \underline{\underline{\vec{p}}} \; a^+(p,\lambda)\, a(p,\lambda)$$

Propagator

$$i D_F(x-y) = \langle 0 | T(\phi(x)\,\phi(y)) | 0 \rangle$$

$$i D_{F\mu\nu}(x-y) = \langle 0 | T(A_\mu(x)\,A_\nu(y)) | 0 \rangle$$

$$= \int \frac{d^4 q}{(2\pi)} \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} e^{-iq\cdot(x-x')}$$

real photons

Coulomb interaction ×✓