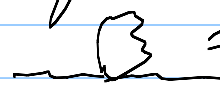
Gauge theory 4 Oct 6

Note Title 10/6/2009

Running coupling

 $+$  $= 0$ N

$$i\Pi^{ab}_{\alpha\beta} = i\left[\frac{11}{3} q_\alpha q_\beta - \frac{19}{6} g_{\alpha\beta}\, q^2\right]\left(\frac{\mu^2}{-q^2}\right)^{\varepsilon/2}\left[-\frac{g^2}{16\pi^2} C_2(adj)\,\delta^{ab}\,\frac{1}{\varepsilon} \cdots\right]$$

Ghosts: 

fermion loop

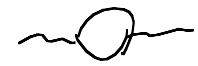
$$i\Pi^{ab}_{\alpha\beta} = -\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2+i\varepsilon}\,\frac{i}{(k-q)^2+i\varepsilon}\left[g\, \underbrace{f^{acd}\, k_\alpha\Big]\Big[g\, f^{bcd}}_{C_2(adj)}(k-q)_\beta\right]$$

$$= i\left[\frac{1}{3} q_\alpha q_\beta + \frac{1}{6} g_{\alpha\beta}\, q^2\right]\left(\quad\right) - - - -$$

Sum is gauge invariant *

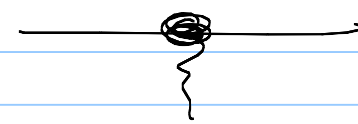
$$\text{Sum} = i\left[q_\alpha q_\beta - g_{\alpha\beta}\, q^2\right]\frac{-g^2}{8\pi^2}\,\frac{5}{3}\, C_2(adj)\,\frac{1}{\varepsilon} \cdots$$

Renormalize at $q^2 = -\mu_R^2$



$$Z_3 = 1 - \frac{g^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{\epsilon} \frac{1}{\epsilon}\left[-\frac{5}{3} C_2(\text{adj}) + \frac{2 n_f}{3}\right]$$

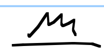
Z_{1F}



Vertex function

QED:

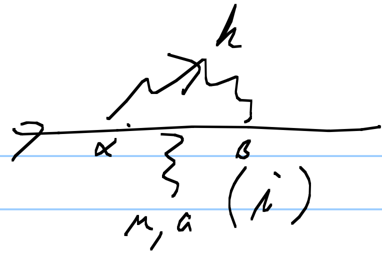
$$e = Z_1^{-1} Z_2 Z_3^{1/2} e_0 \longrightarrow Z_3^{1/2} e_0$$



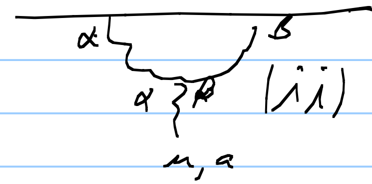
Ward identity $Z_1 = Z_2$

$$\frac{\partial}{\partial p_\mu}\Sigma(p) = \Lambda_\mu(p,p)$$

In YM



(i)



(ii)

i) $-ig\Lambda^a_\mu(p,p') = \left(\frac{-ig}{2}\right)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-i g^{\alpha\beta}}{k^2+i\epsilon} \lambda^b \gamma_\beta \frac{i}{\not p' - \not k + i\epsilon} \gamma_\mu \lambda^a \frac{i}{\not p - \not k + i\epsilon} \lambda^b \gamma_\alpha$

Color factor $\lambda^b \lambda^a \lambda^b = 4\left(C_2(\text{fund}) - \frac{1}{2} C_2(\text{adj})\right) \lambda^a$

Then like QED

$$-ig\Lambda^a_\mu = -ig\gamma_\mu \frac{\lambda^a}{2}\left[\frac{g^2}{8\pi^2}\left(C_2(\text{fund}) - \frac{1}{2} C_2(\text{adj})\right)\right]\left(\frac{\mu^2}{-p^2}\right)^{\epsilon/2}\frac{1}{\epsilon} + \dots$$

renormalize $p^2 = -M_R^2$

off shell, no mass dependence

"mass independent renorm scheme"

ii) $-ig\Lambda^a_\mu(p,p') = ig^3 \int \frac{d^4k}{(2\pi)^4} \left(\gamma^\alpha \frac{\lambda^c}{2}\right) \frac{1}{\not{k}-m} \left(\gamma^\beta \frac{\lambda^b}{2}\right) \frac{f^{abc} N^\mu_{\alpha\beta}}{[(p-k)^2+i\varepsilon][(p'-k)^2+i\varepsilon]}$

$N^\mu_{\alpha\beta} = 2k^\mu g_{\alpha\beta} - g^\mu{}_\alpha k_\beta - g^\mu{}_\beta k_\alpha + \cdots$

Color $f^{abc} \lambda^c \lambda^b = \frac{1}{2} f^{abc} [\lambda^c, \lambda^b] = \frac{1}{2} f^{abc} f^{cbd} 2i \lambda^d$

$= + C_2(\text{adj.}) \lambda^a$

Then

$-ig\Lambda^a_\mu = -ig \frac{\lambda^a}{2} \gamma_\mu \left[\frac{g^3}{8\pi^2} \frac{3}{2} C_2(\text{adj.}) \frac{1}{\varepsilon} + \cdots \right]$

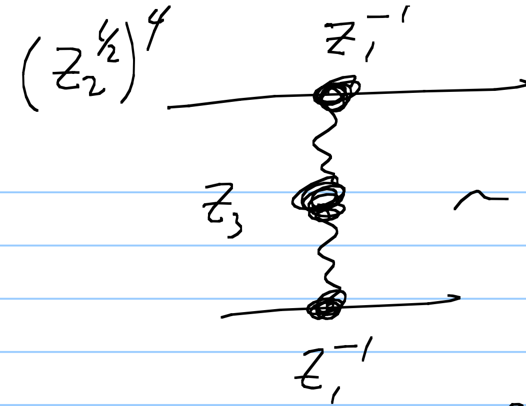
Combine

$Z_{1F} = \left[1 - [C_2(\text{fund}) + C_2(\text{adj.})] \frac{g^3}{8\pi^2} \left(\frac{\mu}{\mu_R}\right)^\varepsilon \frac{1}{\varepsilon} + \cdots \right]$

Lastly 

$\propto \left(\frac{\lambda^a}{2} \frac{\lambda^a}{2}\right) \Rightarrow C_2(\text{fund})\,\mathbb{1}$

$$Z_{2F} = 1 - C_2(\text{fund}) \frac{g^2}{8\pi^2} \cancel{\left(\frac{\mu}{\mu_R}\right)^{\epsilon}} \frac{1}{\epsilon}$$



$\sim g^2$

Combine

$$g = Z_{1F}^{-1} \, Z_2 \, Z_3^{1/2} \, g_0$$

$$= \left(1 + \left(C_2(\text{fund}) + C_2(\text{adj})\right)\frac{1}{\tilde{\varepsilon}}\right)\left(1 - C_2(\text{fund})\frac{1}{\tilde{\varepsilon}}\right)$$

$$\left[1 - \frac{1}{2}\left[-\frac{5}{3} C_2(\text{adj}) + \frac{2}{3} n_f\right]\frac{1}{\tilde{\varepsilon}}\right] g_0$$

$$\frac{1}{\tilde{\varepsilon}} = \frac{g^2}{8\pi^2}\left(\frac{\mu^2}{\mu_R^2}\right)^{\varepsilon/2}\frac{1}{\varepsilon}$$

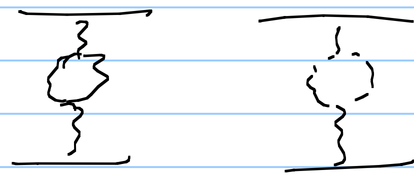
$$= \left[1 - \frac{1}{2}\left[-\frac{11}{3} C_2(\text{adj}) + \frac{2}{3} n_f\right]\frac{1}{\tilde{\varepsilon}}\right] g_0$$

But

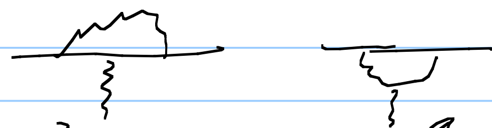
$$\frac{1}{\tilde{\varepsilon}} = \frac{g^2}{8\pi^2}\left[\frac{1}{\varepsilon} + \ln \mu/\mu_R\right] \qquad \mu_R \frac{\partial}{\partial \mu_R}\frac{1}{\tilde{\varepsilon}} = -\frac{g^2}{8\pi^2}$$

$$\mu_R \frac{\partial g}{\partial \mu_R} = -\frac{\overset{**}{g^3}}{16\pi^2}\left[\frac{11}{3}\,\overset{N}{C_2(\text{adj})} - \frac{2}{3} n_f\right] = \beta(g)$$

Of the $\frac{11}{3}N$



$= \frac{5}{3}N$



$\underbrace{-\frac{3}{3}N}$ $\qquad \frac{9}{3}N \Rightarrow \frac{6}{3}N$

all fermions $\Longrightarrow$ + signs

all gauge bosons $\Longrightarrow$ − signs

Running coupling

$$\mu_R \frac{\partial g}{d\mu_R} = -b_0 g^3 \qquad b_0 = \frac{1}{16\pi^2}\left[\frac{11}{3}N - \frac{2}{3}n_f\right]$$

Integrate

$$\int_{g_1}^{g(q^2)} \frac{dg}{g^3} = -b_0 \int_{\mu_1}^{q} \frac{d\mu_R}{\mu_R} = -\frac{1}{2g^2(q^2)} + \frac{1}{2g^2(\mu_1^2)} = -b_0 \ln q/\mu_1$$

Define $\frac{1}{g^2(\mu_1^2)} \equiv 2b_0 \ln \mu_1/\Lambda$ for some Λ

$$\Longrightarrow \frac{1}{g^2(q^2)} = \frac{1}{g^2(\mu_1^2)} + 2\cancel{b}_0 \ln q/\mu_1 = b_0 \ln q^2/\Lambda^2$$

$$g^2(q^2) = \frac{1}{b_0 \ln q^2/\Lambda^2}$$

$$\alpha_N(q^2) = \frac{g^2}{4\pi} = \frac{12\pi}{\underline{(11N - 2n_f)} \ln q^2/\Lambda^2} \quad \not\propto \;\; \propto$$

$q \uparrow \quad \alpha_N \downarrow$

QCD issues $N=3$

1) matching
2) scheme dependence
3) higher order

1) $n_f = ?$

Physics: formulas $q^2 > m_f^2$

$\Pi(q^2) \approx q^2/m^2 \quad q^2 < m^2$

$\ln q^2/\mu^2 \quad q^2 \gg m^2$

$\Rightarrow$ only use quarks lighter than q^2

$\alpha_3(M_Z) \Rightarrow n_f = 5$

$\alpha_3(LHC) \Rightarrow n_f = 6$

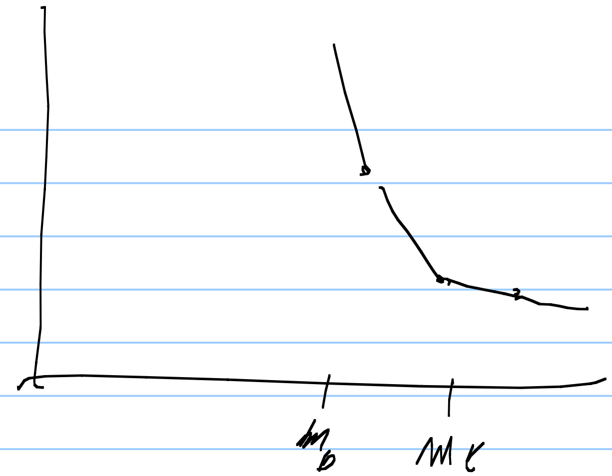
$\alpha_f(4\,GeV) \Rightarrow n_f = 4$

Matching

LHC $(33 - 2 \times 6) = 21$

Z $(\quad\quad 5) = 23$

$(\quad\quad 4) = 25$



$\alpha_s(q^2)$ continuous

$$\alpha_s(m_b) = \frac{12\pi}{23 \ln m_b^2/\Lambda_5^2} = \frac{12\pi}{25 \ln m_b^2/\Lambda_4^2}$$

above — below

Different Λ's in different regions

$$\frac{m_b^2}{\Lambda_4^2} = \left(\frac{m_b}{\Lambda_5^2}\right)^{23/25}$$

$\Longrightarrow$ no such thing as Λ_{QCD}

2) <u>Scheme dependence</u>

<u>MS</u> (minimal subtraction)

$$g_{MS} = \left[1 - b_0 \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \right] g_0$$

$\leftarrow$ only $\frac{1}{\varepsilon}$

0.577

$\overline{MS}$

$$g_{\overline{MS}} = \left[1 - b_0 \frac{g^2}{16\pi^2} \left(\frac{1}{\varepsilon} + \frac{1}{2} \ln 4\pi - \frac{\gamma}{2} \right) \right] g_0$$

$$g_{\overline{MS}} = g_{MS} \left[1 - \frac{g_{MS}^2}{32\pi^2} (\ln 4\pi - \gamma) \right]$$

$$\Longrightarrow \Lambda_{\overline{MS}} \neq \Lambda_{MS}$$

Equivalent to higher order

higher order

$$\alpha_{\overline{MS}}(q^2) = \frac{4\pi}{b_0 \ln q^2/\Lambda_{\overline{MS}}^2} = \frac{4\pi}{b \ln q^2/\Lambda_{\overline{MS}}^2 (1+a)} = \alpha_{\overline{MS}}'(q^2) \doteq \frac{\ln(1+a)}{b_0 [\ln q^2/\Lambda_{\overline{MS}}]^2}$$

Next order β function ← 2 loops

$$\mu_R \frac{\partial g}{\partial \mu_R} = -b_0 g^3 + b_1 g^5$$

Solve

$$x = g^2$$

$$-\int_{\mu_1}^{q} b_0 \frac{d\mu_R}{\mu_R} = \int \frac{dg}{g^3 + b_1/b_0\, g^5} = \frac{1}{2}\int \frac{dx}{x^2(1 + b_1/b_0\, x)} = \frac{1}{2}\int dx \left(\frac{1}{x^2} - \frac{b_1}{b_0}\frac{1}{x}\right)$$

$$2b_0 \ln q/\mu_R = \underset{\substack{\uparrow \\ 1/g^2}}{\frac{1}{x}} - \frac{1}{x_1} - \frac{b_1}{b_0} \ln \underset{\substack{\uparrow \\ g^2}}{x}/x_1{}_2$$

$$\frac{1}{x_1} + \frac{b_1}{b_0} \ln x_1 = b_0 \ln \mu^2/\Lambda^2$$

$$g^2 = \frac{1}{b_0 \ln q^2/\Lambda^2 \left(1 + \frac{b_1}{b_0} \frac{\ln g^2}{\ln q^2/\Lambda^2}\right)} = \frac{1}{b_0 \ln q^2/\Lambda^2} \left(1 - \frac{b_1}{b_0^2} \frac{\ln \ln q^2/\Lambda^2}{\ln q^2/\Lambda^2}\right)$$

In QCD

$$\alpha_s(q^2) = \frac{12}{(33 - 2n_f) \ln q^2/\Lambda^2} \left[1 - \frac{6(153 - 19n_f)}{(33 - 2n_f)^2} \frac{\ln(\ln q^2/\Lambda^2)}{\ln(q^2/\Lambda^2)}\right]$$