QFT II 9/15

Note Title 9/15/2009

Heat Kernal

$\swarrow$ more Traces

Recall $\det \mathcal{O} = e^{\mathrm{Tr} \ln \mathcal{O}} = e^{\int d^4x \langle x | \ln \mathcal{O} | x \rangle}$

Use $\ln b/a = \int_0^\infty \frac{dx}{x} \left(e^{-ax} - e^{-bx} \right)$

drop

$\Rightarrow \langle x | \ln \mathcal{O} | x \rangle = -\int_0^\infty \frac{d\tau}{\tau} \langle x | e^{-\tau \mathcal{O}} | x \rangle$ + const

Heat Kernal

$H(x,\tau) \equiv \langle x | e^{-\tau \mathcal{O}} | x \rangle$

Math Gilkey , DDH

$$\mathcal{O} = d_\mu d^\mu + m^2 + \sigma(x) \qquad d_\mu = \partial_\mu + \Gamma_\mu(x)$$

$$H(x,\tau) = \frac{1}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \dots \right]$$

$a_0 = 1$

$a_1 = -\sigma$

$a_2 = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]]$

Then

$$\langle x | \ln \mathcal{O} | x \rangle = \frac{-i}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} m^{d-2n} \Gamma(n - d/2)\, a_n(x)$$

Example QED

$$d_\mu = \partial_\mu + ie A_\mu \qquad , \quad \sigma = 0$$

$$[d_\mu, d_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ie F_{\mu\nu}$$

$$a_2 = \frac{(ie)^2}{12} F_{\mu\nu} F^{\mu\nu}$$

$$\langle x | \ln \mathcal{O} | x \rangle = \frac{-i}{(4\pi)^{d/2}} m^{d-4} \frac{(ie)^2}{12} F_{\mu\nu} F^{\mu\nu} \; \Gamma(2 - d/2) = i\, \Pi(0) F_{\mu\nu} F^{\mu\nu}$$

– no need to calculate loops!

Fermions in Path Integrals

Bosons

$$Z_0[J] = N\int[d\phi]\, e^{i\int d^4x[\mathcal{L}_0 + J\phi]} = N\, e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)}$$

<u>Fermion</u> $\psi, \bar{\psi}$

– sources $\eta(x), \bar{\eta}(x)$

$$Z_0[\eta, \bar{\eta}] = N\, e^{-i\int d^4x\, d^4y\, \bar{\eta}(x)\, S_F(x-y)\, \eta(y)} = N\int d\psi\, d\bar{\psi}\, e^{i\int d^4x[\mathcal{L}_0 + \bar{\eta}\psi + \bar{\psi}\eta]}$$

Issues ordering + signs + $\eta, \bar{\eta}$ will be <u>anticommuting</u>

$$G^{(2)}(x, x') = \frac{1}{Z[0]}(+i)(-i)\frac{\delta}{\delta\bar{\eta}(x)\,\delta\eta(x')} Z[\eta, \eta'] = \langle 0 | T\, \psi(x)\, \bar{\psi}(x') | 0 \rangle$$

$$= i S_F(x-x')$$

Anticommuting sources

$$\frac{\delta}{\delta \bar\eta(x)} \int d^4y \; \bar\eta(y)\psi(y) = \psi(x)$$

$$\frac{\delta}{\delta \eta(x)} \int d^4y \; \bar\psi(y)\eta(y) = -\bar\psi(x)$$

Then

$$\frac{\delta}{\delta \bar\eta(x)} e^{i\int d^4y[\bar\eta\psi + \bar\psi\eta]} = i\psi(x)\, e^{i\,\cdots}$$

$$\frac{\delta}{\delta \eta(x)} \quad " \quad = -i\bar\psi(x)$$

Pert Theory

$$Z[\eta,\eta'] = \int d\psi\, d\bar\psi\; e^{i\int d^4x\,[\mathcal{L}_0 + \mathcal{L}_I(\psi,\bar\psi) + \bar\eta\psi + \bar\psi\eta]}$$

$$= e^{i\int d^4y\, \mathcal{L}_I\left(-i\frac{\delta}{\delta\bar\eta},\, +i\frac{\delta}{\delta\eta}\right)} Z_0[\eta,\bar\eta]$$

$\uparrow$ expand + Feynman rules

Grassmann numbers

$$\{\alpha, \alpha\} = 0 \qquad \alpha^2 = 0$$

$$\{\alpha, \beta\} = 0$$

$$f(\alpha) = f_0 + f_1 \alpha$$

$$g(\alpha, \beta) = g_0 + g_1 \alpha + g_2 \beta + g_3 \alpha \beta$$

Diff.

$$\frac{d}{d\alpha} \alpha = 1 \quad , \quad \frac{d}{d\alpha} \beta = 0$$

$$\Longrightarrow \frac{d}{d\alpha} f(\alpha) = f_1 \quad ,$$

$$\frac{d}{d\alpha} g(\alpha,\beta) = g_1 + g_3 \beta \quad ; \quad \frac{d}{d\beta} g(\alpha,\beta) = g_2 - g_3 \alpha$$

Integration

translation property $\quad \int d\alpha \; f(\alpha) = \int d\alpha \, f(\alpha+\beta)$

$$f(\alpha+\beta) = f_0 + f_1\alpha + f_1\beta$$

$$\int d\alpha \left(f_0 + f_1\alpha\right) = \int d\alpha \left(f_0 + f_1\alpha + f_1\beta\right)$$

$$\Rightarrow \int d\alpha \; f_1\beta = 0 \implies \int d\alpha = 0$$

$$\int d\alpha \; \alpha \equiv \underline{1}$$

$$\int d\alpha \; f(\alpha) = f_1$$

More variables α_1, $\bar{\alpha}_1$

$$\int d\bar{\alpha}_1 \, d\alpha_1 \, e^{-\bar{\alpha}_1 m \alpha_1} = \int d\bar{\alpha}_1 \, d\alpha_1 \left(1 - \bar{\alpha}_1 m \alpha_1\right) = m$$

<u>Yet more</u> $(\alpha_1, \alpha_2 \, \cdots)(\bar{\alpha}_1, \bar{\alpha}_2, \cdots)$

$$\int d\bar{\alpha}_N \cdots d\bar{\alpha}_1 \, d\alpha_N \cdots d\alpha_1 \, e^{-\bar{\alpha}_i M_{ij} \alpha_j} \xrightarrow[\text{+ use above}]{\text{diagonalize}} \prod_{i=1}^{N} m_i = \det M$$

m_i ← eigenvalues

<u>Fields</u> $\alpha_i \to \psi(x)$ $\quad \bar{\alpha}_i \to \bar{\psi}(x)$

$$Z = \int d\bar{\psi} \, d\psi \, e^{i \int d^d x \, \bar{\psi}(x) \, \mathcal{O} \, \psi(x)} = \det \mathcal{O}$$

↑ Boson $[\det \mathcal{O}]^{-1/2}$

Sources

$$Z[\eta,\bar\eta] = \int [d\bar\psi][d\psi]\, e^{i\int d^4x\left[\bar\psi(i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta\right]}$$

Complete square

$$\psi'(x) = \psi(x) + \int d^4y\; S_F(x-y)\,\eta(y) \qquad (i\not\partial - m)S_F(x-y) = \delta^4(x-y)$$

$$\bar\psi'(x) = \bar\psi(x) + \int d^4y\; \bar\eta(y)\,S_F(x-y)$$

The $\int d^4x\, \bar\psi'(i\not\partial - m)\psi' = \int d^4x\left[\bar\psi(i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta\right] + \int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\eta(y)$

use $\int d\psi = \int d\psi'$

$$Z[\eta,\bar\eta] = Z[0,0]\, e^{-i\int d^4x\, d^4y\; \bar\eta(x)\, S_F(x-y)\, \eta(y)}$$

Spin Statistics Thm

Integer spin $\Rightarrow$ commutators $\Rightarrow$ Bose Einstein

Half integral $\Rightarrow$ anticommutators $\Rightarrow$ F.D statistics

Proof: Pauli, Book Streater + Whiteman, Pedagogic = Weinberg QFT I

Ingredients: Lorentz Invariance
positive energies
positive norms
causality

In 2D anyons $|\psi_1 \psi_2\rangle = e^{i\theta} |\psi_2 \psi_1\rangle$ any θ

Dirac Algebra

Fermions $= -ie\gamma_\mu$

$$\frac{i}{\not{p}-m+i\varepsilon} = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

1) Standard Basis

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

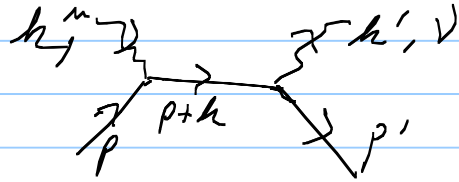
Basis 4x4 matrices 16 matrices as a basis

$$1, \quad \gamma_5, \quad \gamma_\mu, \quad \gamma_\mu\gamma_5, \quad \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\# \quad 1, \quad 1, \quad 4, \quad 4, \quad \underbrace{0i, (12, 13, 23)}_{6} \quad = 16$$

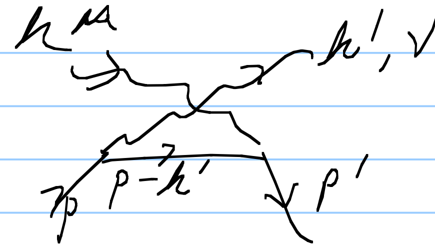
$$\sigma_{\mu\nu}\gamma_5 = -i\,\varepsilon_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta}$$

2) Order matters



(a)

+



(b)

$$\mathcal{M}_a = \bar{u}(p')\left[-ie\gamma_\nu \frac{i}{\not p+\not k-m} -ie\gamma^\mu\right] u(p) \quad \varepsilon^*_\nu(k') \quad \varepsilon_\mu(k)$$

$$\mathcal{M}_b \quad \bar{u}(p')\left[-ie\gamma_\mu \frac{i}{\not p-\not k'-m} -ie\gamma^\nu\right] u(p) \quad \varepsilon^*_\nu(k') \quad \varepsilon_\mu(k)$$

$p'+k-m$

Sample calculation – Gauge Invariance

$$\mathcal{M} = \varepsilon^*_\nu(k')\,\varepsilon_\mu(k)\;\mathcal{M}^{\mu\nu}$$

Since $A^\mu \to A^\mu + \partial^\mu \Lambda \Rightarrow$ gauge invariance

$$\varepsilon^\mu(k) \to \varepsilon^\mu(k) + a\,k^\mu \quad \nearrow$$

Requires $k^\mu \mathcal{M}_{\mu\nu} = 0$ and $k'^\nu \mathcal{M}_{\mu\nu} = 0$

Check

$$k^\mu \mathcal{M}_{\mu\nu} = (-ie)^2 i\,\bar{u}(p')\left[\gamma_\nu \frac{1}{\not p + \not k - m}\not k + \not k \frac{1}{\not p' - \not k - m}\gamma_\nu\right]u(p)$$

Use $\not k\, u(p) = (\not p + \not k - m)\,u(p)$, $\quad \bar{u}(p')\not k = \bar{u}(p')\left[-(\not p' - \not k - m)\right]$

$$\Rightarrow k^\mu \mathcal{M}_{\mu\nu} = (\;)\,\bar{u}\left[\gamma_\nu - \gamma_\nu\right]u(p) = 0 \quad \checkmark$$