Pert theory - functional

\[ e^{iS_{\phi}}\left[ \mathcal{L}_0 + \mathcal{L}_I(\phi,\phi^*) + \mathcal{J} \phi + \mathcal{J}^* \phi^* \right] \]

\[ e^{iS_{\phi}}\mathcal{L}_I(\phi,\phi^*) \quad e^{iS_{\phi}}\left[ \mathcal{L}_0 + \mathcal{J} \phi + \mathcal{J}^* \phi^* \right] \]

\[ = e \quad e^{iS_{\phi} \mathcal{L}_I(-i\frac{\delta}{\delta \phi^*},-i\frac{\delta}{\delta \phi})} \quad e^{iS_{\phi}}\left[ \mathcal{L}_0 + \mathcal{J} \phi + \mathcal{J}^* \phi^* \right] \]

Then

\[ S[\phi(\lambda)] \\ e^{iS_{\phi}}\left[ \mathcal{L}_0 + \mathcal{L}_I(\phi,\phi^*) + \mathcal{J} \phi + \mathcal{J}^* \phi^* \right] \]

\[ = e \quad iS_{\phi} \mathcal{L}_I(-i\frac{\delta}{\delta \phi^*},-i\frac{\delta}{\delta \phi}) \quad N \quad e \]

No external \( \phi \Rightarrow J = 0 \) after diff.

\[ \frac{\delta}{\delta \phi^*} \frac{\delta}{\delta \phi} \rightarrow iD \phi \text{ in pair} \]
\[ z \left[ J_m \right] = \text{Sech}A \left( 1 + i \int \text{Sech}^2 A \text{e}^z A^m \text{df}(y, y) \right) \]

\[ - \frac{1}{2} \text{Sech}^2 A (y-x) \text{An}(z)(y-x) \text{An}(z) \left[ \text{df}(y, x) \text{df}(z, y) \text{df}(z, x) \right] + \cdots \]

\[ \times \exp \int \text{Sech}^2 \left[ - \frac{i}{2} E_y E_x - i A^m \right] \]

Use functional trick
\[ A_j(x) \text{ e}^{-i \frac{\delta}{\delta A_j(x)}} \]

Do A integral
\[ z \left[ J_m \right] = \left( 1 + i \int \text{Sech}^2 A \text{e}^z \text{df}(y, y) \frac{\delta}{\delta A_j(x)} \frac{\delta}{\delta A_j(x)} \right) \int_{\text{J=0}} \]

Finally
\[ G_{\mu \nu}^{(2)}(x, y) = -i \frac{\delta^2}{\delta \overline{J}_{\mu}(x) \delta \overline{J}_{\nu}(y)} \left. \right|_{J=0} \]

\[ = \sum_{\text{max}} + \sum_{\text{max}} + \sum_{\text{max}} + \sum_{\text{max}} \]

\[ i D_{\text{max}}^{\text{max}} (N_1, -N_2) \]
\[ \text{Use } \quad i S^{\phi}[\Phi] e^{-i\Phi / 2} = N \left[ \det \mathbf{O} \right]^{1/2} \]

\[ \text{Vac. Pol} \]

\[ Z[J, \bar{J}] = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{i S^{\phi}[\Phi + \mathbf{A}^a + \mathbf{A}^a + \mathbf{A}^a \bar{\phi} + \bar{\phi} \mathbf{A}]} \]

no external \( \phi \) \( \Rightarrow \) set \( J = 0 \)

\[ = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{i S^{\phi}[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}]} \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \]
\[ I(\phi, \phi^*, A) = (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi \]

\[ = (\partial^2 - i e A^\mu) \phi^* (\partial_\mu + i e A_\mu) \phi - m^2 \phi^* \phi \]

\[ \text{int. by parts} \]

\[ = - \phi^* (\partial^2 + i e A^\mu) (\partial_\mu + i e A_\mu) \phi - m^2 \phi^* \phi \]

\[ = - \phi^* (D_\mu D^\mu + m^2) \phi \implies - \phi^* \nabla^2 \phi \]

\[ \int d\phi \, d\phi^* e^{-i S(\phi, \phi^*, \partial^2 + m^2)} \phi = N [\text{det} (\partial^2 + m^2)]^{-1} e^{-\frac{m^4}{2}} \]

\[ = e^{i W[A]} \]
\[ L(\phi, \phi^*, \Lambda) = (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi^* \phi \]
\[ = (\partial^\mu - i e A^\mu) \phi^* (\partial_\mu + i e A_\mu) \phi - m^2 \phi^* \phi \]
\[ \quad \text{(integrate over}\ \partial \text{)} \]
\[ = - \phi^* (\partial^\mu + i e A^\mu) (\partial_\mu + i e A_\mu) \phi - m^2 \phi^* \phi \]
\[ = - \phi^* (D^2 + m^2) \phi \quad \Rightarrow \quad - \phi^* \Delta \phi \]

\[ S d\phi d\phi^* e^{-i S d\phi^* \phi^* (D^2 + m^2) \phi} = N [\text{det} (D^2 + m^2)]^{-1} e^{-\frac{1}{2} \text{WCA}} \]
\[ \left[ \text{det} \left( D^2 + m^2 \right) \right]^{-1} = \mathbb{I} - \text{SAdv} \left< \phi_1 \ln \left( D^2 + m^2 \right) \phi_1 \right> \]

**Pert Theory**

\( O_0 = (\Box + m^2) \)

\( D^2 + m^2 = \Box + m^2 + V = O_0 \left( O_0^{-1} (D^2 + m^2) \right) \)

\( O_0^{-1} = D_f \)

\( = O_0 \left( 1 + O_0^{-1} V \right) \)

\[ \text{det} \left( D^2 + m^2 \right)^{-1} = \text{det} O_0^{-1} \left( 1 + O_0^{-1} V \right)^{-1} = N \text{det} \left( 1 + D_f V \right)^{-1} \]

\[ = N \left< \phi_1 \ln \left( 1 + D_f V \right) \phi_1 \right> \]

\[ = N \mathbb{I} - i \text{SAdv} \left[ <\phi_1 | D_f \phi_1 > \frac{1}{2} <\phi_1 | D_f V D_f V \phi_1 > \right] \]
\[ D^2 = \partial^2 + i e A_m \partial^m + i \frac{e^2}{2} A^m A_m - e^2 A_m A^m \]

Look for \( O(e^2) \)

\[ \langle N! D_F V | N \rangle = -e^2 A_m A^m \langle N \rangle D_F (x-x) \]

\[ \text{Say } \langle N! D_F V | N \rangle \langle N! D_F V | N \rangle = \text{Sd}_{N,N} D_F (x-x) (A_m A^m + 2A^m A_m) D_F (y-x) (A_m A^m + 2A^m A_m) \]

\[ e^{i W[A_m]} = e^{i \left\{ d^m \partial^m (x-x \partial^m A^m + \frac{1}{2} \int \text{d}^4 x \partial^m \left( e^{2A_m A^m + A_m A^m} \partial^m A_m - A_m A^m \right) D_F (y-x) \right\}} \]

\[ = \langle N \rangle + \langle O \rangle \]
\[ Z[\bar{\phi}] = \int dA e^{i \sum_{\gamma} \left[ -\frac{1}{4} F^2 + \text{Im} A \gamma^4 \right] + i \text{WCA}} \]

Renormalization + Effective I

- Large mass

\[ \Pi^\gamma_{\mu\nu}(q^2) = (g^\gamma_{\mu} g^\nu_{\nu} - g^\gamma_{\mu} g^\nu_{\nu}) \Pi(q^2) \]

\[ \Pi(q^2) = \Pi(0) + \frac{\alpha}{\epsilon^2} \frac{q^2}{m^2} \]

In matrix element

\[ i \langle \bar{\psi} \gamma \lambda | A | \psi \rangle \rightarrow g^\gamma_{\mu} \lambda \]

\[ i W[A] \rightarrow \int d\Sigma A_\mu \left( \partial^\nu \delta^\gamma_{\nu} - g_{\mu\nu} D^\nu \right) A_\nu \Pi(0) \]

\[ e = e - \frac{\alpha}{120 \pi} A_\mu \left( \partial^\nu \delta^\gamma_{\nu} - g_{\mu\nu} D^\nu \right) A_\nu A_{\lambda} \]
\[ \frac{1}{2} A_\mu \left( \square g^{\mu \nu} - \partial^\mu \partial^\nu \right) A_\nu = -\frac{1}{4} E_{\mu \nu} F^{\mu \nu} \]

Overall

\[ Z[J] = \mathcal{S}[DA] \cdot \text{Sdet}\left[ -\frac{1}{4} E_{\mu \nu} F^{\mu \nu} (1 - \Pi / \alpha) \right] - \frac{1}{128 \pi} E_{\mu \nu} D^2 F^{\mu \nu} \]

Two results

1) Renormalization

\[ A_n \rightarrow A_n Z_3^{1/2} \quad \text{with} \quad Z_3 = \frac{1}{1 - \Pi / \alpha} \quad \text{and} \quad N[DA] = N[DA'] \]

2) Effective \( Z \)

\[ -\frac{1}{128 \pi} E_{\mu \nu} D^2 F^{\mu \nu} \]

In part theory

\[ \alpha \rightarrow \alpha \left[ \frac{\alpha}{1 - \Pi / \alpha} \right] (g^{\mu \nu} g^{\mu \nu} - g^{\mu \nu} g^{\mu \nu}) \]

\[ \Theta \quad + \quad \frac{\delta}{\delta \alpha} \]
Appelquist-Darragh-Thoo

- Heavy fields
  1) renormal constant
  2) suppressed by $\frac{1}{M^n}$

Note: if $m^2 << g^2$ cannot use eff $I$

$$\Pi (g^2) = \frac{\alpha}{12\pi} \ln \frac{g^2}{m^2}$$

- $\ln \Box$ ill defined
- $\ln - g^2 = \ln (g^4 + i \Pi \Theta (g^2))$  - eff $I$ has to be real

Physical reason
- heavy mass non-local $\Rightarrow$ local $\bar{\psi}$
- mass light $\not\Rightarrow$ not local $\Rightarrow$ non-local $\bar{\psi}$