Basics 3

Note Title ~~1/22/2010~~ 1/26/10

Fund. expt. fact $\qquad E = \hbar\omega$

$$[q, p] \longrightarrow [\phi(x,t), \pi(x',t)] = i\hbar\,\delta^3(\vec{x}-\vec{x}')$$

$$\Longrightarrow [a(\vec{p}), a^+(\vec{p}\,')] = (2\pi\hbar)^3\,\delta^3(\vec{p}-\vec{p}\,')$$

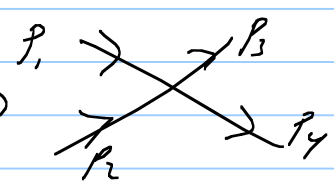
$$\Longrightarrow H = \int \frac{d^3p}{(2\pi\hbar)^3}\,\hbar\omega_p\, a^+(p)\,a(p) = \sum_s \hbar\omega_s\, a^+(s)\,a(s)$$

Could have reversed it $\qquad$ quanta $\rightarrow H = \sum \hbar\omega\, a^+ a \rightarrow [a, a^+]$

$\rightarrow [\phi, \pi]$

<u>a, a^+ bookkeeping devices</u>

$$\langle p_3\, p_4 \,|\, \phi^4 \,|\, p_1, p_2\rangle \Longrightarrow$$



Inverting

Recall q.m.

$$x = \sqrt{\frac{\hbar}{2\omega}}\,(a + a^{+}) \quad , \quad p = -i\sqrt{\frac{\hbar\omega}{2}}\,(a - a^{+})$$

$$a = \sqrt{\frac{\omega}{2\hbar}}\left(x + i\frac{p}{\omega}\right) \quad , \quad a^{+} = \sqrt{\frac{\omega}{2\hbar}}\left(x - i\frac{p}{\omega}\right)$$

$$[a, a^{+}] = 1$$

$\nwarrow [x, p]$

$\swarrow \phi(x,t) = \int d^3k\, N_k \left(a \;\ldots\; a^{+}\right)$

$\leftarrow \frac{1}{N^2}\phi$

Fields

$$a(k) = \sqrt{\frac{\omega}{2\hbar}} \int d^3x \; e^{i(\omega_k t - \vec{k}\cdot\vec{x})}\left[\phi(x,t) + \frac{i}{\omega_k}\,\Pi(x,t)\right]$$

$$= \frac{1}{2}\left[\underset{\uparrow \phi}{a(k)} + \underset{\uparrow \Pi}{a(k)} + e^{i2\omega t}\left(\underset{\uparrow \phi}{a^{+}(-k)} - \underset{\uparrow \Pi}{a^{+}(-k)}\right)\right] = a(k)$$

$$[a(k), a^\dagger(k')] = NN' \int d^3x \, e^{i(\ )} \int d^3x' \, e^{i(\ )'} \underbrace{[\phi + \tfrac{i}{\omega}\pi, (\phi - \tfrac{i}{\omega}\pi)']}_{\delta^3(x-x')}$$

$$= (2\pi\hbar)^3 \delta^3(k-k')$$

$[\phi, \pi] \Longrightarrow [a, a^\dagger]$ required

Conservation of Energy

$$H = \int d^3x \left[\pi \dot{\phi} - \mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi) \right]$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$\frac{dH}{dt} = \int d^3x \left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)\dot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \underbrace{\frac{d}{dt}\dot{\phi}}_{\ddot{\phi}} - \underbrace{\frac{d}{dt}\mathcal{L}(\phi, \dot{\phi}, \nabla\phi)}_{-\left(\frac{\partial \mathcal{L}}{\partial \phi}\dot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\ddot{\phi} + \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi}\frac{d}{dt}\vec{\nabla}\phi\right)} \right]$$

cancel

$$= \int d^3x \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial(\nabla\phi)} - \frac{\partial \mathcal{L}}{\partial \phi} \right] \dot{\phi}$$

$$- \int d\vec{A} \cdot \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \times \dot{\phi}$$

If surface at ∞ & fields vanish

$\Rightarrow$ Surface term $= 0$

$[\quad] \Rightarrow$ E.L. eq $\Rightarrow \frac{dH}{dt} = 0$ energy conserved

Extra information

finite surface $\frac{dH}{dt}$ = flow of energy across surface

$$\left\{ \boxed{\begin{aligned} \vec{P} &= \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \dot{\phi} \\ \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \end{aligned}} \right.$$

$\Leftarrow$ flow of energy = "momentum density"

$E \sim mc^2$

$\vec{P} = m\vec{v}$

Differential conservation law

$$\frac{d\mathcal{H}}{dt} + \vec{\nabla}\cdot\vec{P} = 0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}\right) + \vec{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\nabla\phi}\dot\phi = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi}\dot\phi + \frac{\partial\mathcal{L}}{\partial\dot\phi}\ddot\phi + \frac{\partial\mathcal{L}}{\partial\nabla\phi}\nabla\dot\phi$$

Conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

charge density
energy density

flow of charge
energy

Total momentum

string, or 3d

$$\phi = \int \frac{d^3k}{(2\pi)^3}\left[e^{i(\omega t - k\cdot x)} a(k) + \dots \right]$$

$$\vec{P} = \int d^3x \; \frac{\partial \mathcal{L}}{\partial \nabla \phi}\, \dot{\phi} = \underline{\int d^3x \, (\vec{\nabla}\phi)\left(\frac{\partial}{\partial t}\phi\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2\omega} \propto \omega\, \vec{p} \left[a(p)\, a^\dagger(p) + a^\dagger(p)\, a(p) + \underbrace{e^{2i\omega t} a^\dagger(p)\, a^\dagger(-p) + e^{-2i\omega t} a(p)\, a(-p)}_{\int d^3p \; p\,[\;] \,=\, 0 \text{ by symmetry } p \to -p} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\; \vec{p}\; a^\dagger(p)\, a(p)$$

use $\int d^3p \, \vec{p} = 0$

Then

$$\vec{P}\, |\vec{p}_1\rangle = \int \frac{d^3p}{(2\pi)^3}\; \vec{p}\; a^\dagger(p)\, \underbrace{a(p) \; a^\dagger(p_1)|0\rangle}_{(\underbrace{[a, a^\dagger]}_{\delta^3(p - p_1)} + a^\dagger \underbrace{a}_{0})|0\rangle} = \vec{p}_1\, |\vec{p}_1\rangle \quad ✓$$

Enough! $\hbar = c = 1$ units

- clears up notation $\Rightarrow$ see important part.
- read all QFT books

1) $c = 1$ (t -years, x = light years)

$$\left.\begin{array}{l}(ct, \vec{x}) = (t, \vec{x}) \\ E = \sqrt{p^2c^2 + m^2c^4} \rightarrow \sqrt{p^2 + m^2}\end{array}\right\} \begin{array}{l} t, x \\ E, p, m \end{array}$$

2) $\hbar = 1$ $\quad \Delta E \, \Delta t \geq \hbar \sim 1$

units of $\Delta t \sim \frac{1}{\Delta E}$, $\Delta x \sim 1/\Delta p$

$$e^{-i\left(\frac{Et - p\cdot\vec{x}}{\hbar}\right)} \Rightarrow e^{-i(Et - p\cdot x)} \Leftarrow \text{dimensionless}$$

3) E+M = Heaviside Lorentz units

$$\alpha = \frac{1}{137} = \frac{e^2}{4\pi\hbar c} \rightarrow \frac{e^2}{4\pi}$$

4) <u>Conversion</u>

$$\hbar c = 197 \text{ MeV-fm} = 1970 \text{ eV Å}$$

$$c = 3 \times 10^8 \text{ m/sec}$$

4 vectors

$\mu = 0 \Rightarrow$ time

$$x^\mu = (ct, \vec{x}) \to (t, \vec{x})$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) \to (\partial_t, -\vec{\nabla}) = (\partial_0, -\nabla)$$

$$p^\mu = (E/c, \vec{p}) \to (E, \vec{p})$$

Invariants

$$x_\mu x^\mu = t^2 - \vec{x}$$

$$p_\mu p^\mu = E^2 - \vec{p}^{\,2} = m^2$$

$$E^2 = p^2 + m^2$$

$$p_\mu x^\mu = Et - \vec{p}\cdot\vec{x} = p \cdot x$$

$$e^{-ip\cdot x} = e^{-i(Et - \vec{p}\cdot\vec{x})}$$

String $\partial^\mu = \left(\frac{1}{v} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x} \right)$

$$\mathcal{L} = \frac{1}{2} \frac{1}{v^2} \dot{\phi}^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi = \frac{1}{2} \left[\left(\frac{1}{v} \frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 \right]$$

Eq of motion $\left[\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right] \phi = \partial_\mu \partial^\mu \phi \equiv \Box \phi$

Integration

$\int dt \, d^3x \equiv \int d^4x$ ← relativistic invariant

Lorentz trans $(\vec{x}, t) \to (x', t')$

$J = 1$

Action principle

$$S = \int dt\, d^3x\, \mathcal{L}(\phi, \partial_0\phi, \vec{\nabla}\phi) \overset{\text{rel. inv}}{=} \int d^4x\, \mathcal{L}(\phi, \partial^\mu\phi)$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\dot\phi)} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 = \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi)} - \frac{\partial \mathcal{L}}{\partial\phi}$$

Rel. inv notation: $\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

Example string

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\, \partial^\mu\phi), \quad \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = \partial_\mu\phi \Rightarrow \partial^\mu\partial_\mu\phi = 0 = \Box\phi \checkmark$$

Dimensional analysis

H = energy $\sim E$, $\cancel{\hbar}$

$t \sim 1/E$

$S = \int dt\, L$ = dimensionless $\sim E^0$

$= \int \underbrace{dt\, d^3x}_{1/E^4}\, \mathcal{L} \leftarrow (E)^4$ called $d=4$

$\mathcal{L} = \frac{1}{2} \frac{\partial\phi}{\partial t} \frac{\partial\phi}{\partial t}$ $\frac{\partial}{\partial t}, \partial_\mu \sim E$ $\partial_\mu e^{-ip\cdot x} = -ip_\mu e^{-ip\cdot x}$

$$\boxed{\begin{aligned} \partial^\mu &\sim E^1 \\ \phi &\sim E^1 \end{aligned}}$$

, $\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} \sim E^2$, $[\phi, \pi] = i\,\delta^3(x-x') \sim \frac{1}{x^3} \sim E^3$