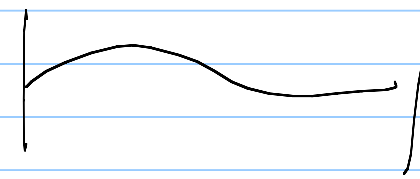
Basics 2

Note Title 1/21/2010

$$S = \int dt \int dx \, \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 \right]$$



$$\star \quad [\phi(x,t), \pi(x',t)] = i\hbar\, \delta(x-x') \qquad \pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot\phi}$$

$$\phi(x,t) = \sum_s N_s \left[e^{-i(\omega_s t - k_s x)} a(s) + e^{+i(\quad)} a^\dagger(s) \right]$$

$$[a(s), a^\dagger(s')] = \delta_{ss'}$$

$$N_s = \sqrt{\frac{\hbar v^2}{2\omega_s L}}$$

Hamiltonian

$$H = \int dx \ \frac{1}{2}\left[\frac{1}{N^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right]$$

$$= \int dx \ \frac{1}{2}\sum_{s,s'} N_s N_{s'}\left[\frac{1}{N^2}(-i\omega_s)(-i\omega_{s'})\left(e^{-i(\ )} a(s) - e^{+i(\ )} a^{+}\right)\left(e^{-i(\ )} a(s')\qquad\right)\right.$$

$$\left.(ik_s\, ik_{s'})\left(\qquad\right)\left(\qquad\right)\right]$$

$$\int dx\, e^{ik_s x} e^{-ik_{s'} x} = L\,\delta_{k_s, k_{s'}}$$

$$\int dx\, e^{ik_s x} e^{ik_{s'} x} = L\,\delta_{k_s, -k_{s'}}$$

$$\omega_{s'} = \omega_s$$

$$\omega_s = |k_s| N$$

$$H = \frac{1}{2}\sum_s \frac{\hbar N^2}{2\omega_s L} \times L\left\{\left(\frac{-\omega_s^2}{N^2} + k_s^2\right)\left(e^{-2i\omega t} a(s)\, a(-s) + e^{+2i\omega t} a^{+}(s)\, a^{+}(-s)\right)\right.$$

$$\left. + \left(\frac{\omega_s^2}{N^2} + k_s^2\right)\left(a(s)\, a^{+}(s) + a^{+}(s)\, a(s)\right)\right\}.$$

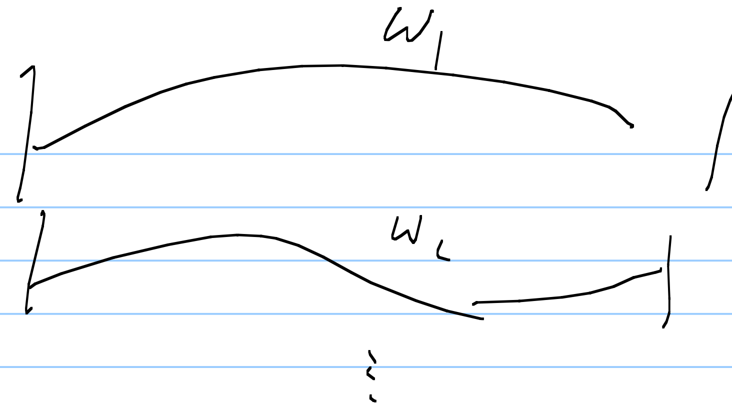
$$H = \sum_s \hbar \omega_s \frac{1}{2}\left(a(s)\, a^+(s) + a^+(s)\, a(s) \right)$$

$$= \sum_s \hbar \omega \Big[\underbrace{a^+(s)\, a(s)}_{\text{Number operator}} + \frac{1}{2} \Big] = H_0 + \underbrace{E_0}_{\text{Zero point energy}}$$

$$\sum_s \frac{1}{2} \hbar \omega_s$$

$$H_0 = \sum_s \hbar \omega_s \, a^+(s)\, a(s)$$

Spectrum



Lowest state

$$|0,0,0,\cdots\rangle \equiv |0\rangle$$

no w_1, no w_2

$$H_0|0\rangle = 0 \Rightarrow a(s)|0\rangle$$

$$a^\dagger(s)\,a(s)\,|0\rangle = 0$$

Single quanta

$$|1,0,0,0\cdots\rangle = a^\dagger(1)|0\rangle$$

$$H_0|1,0\cdots 0\rangle = \sum_s \hbar\omega_s\, a^\dagger(s)\,a(s)\; a^\dagger(1)|0\rangle$$

$$= \hbar\omega_1\, a^\dagger(1)\,\overbrace{a(1)\,a^\dagger(1)}\,|0\rangle = \hbar\omega_1\, a^\dagger(1)|0\rangle$$

$$\Big(\underbrace{[a,a^\dagger]}_{1} + \cancel{a^\dagger a}^{\,0}\Big)|0\rangle$$

Multi quanta

$|2,0,0\,0\rangle = \frac{1}{\sqrt{2}}\, a^{+}(1)\, a^{+}(1)\, |0\rangle \quad \Rightarrow H_0 |\;\rangle = 2\hbar\omega |\;\rangle$

$|1,1,0\,0\,0\rangle = a^{+}(1)\, a^{+}(2)\, |0\rangle \quad \Rightarrow H_0 |\;\rangle (\hbar\omega_1 + \hbar\omega_2) |\;\rangle$

$\Longrightarrow$ know all the states ("Fock space")

Why Equal time Commutators?

$$[\phi(x,t), \Pi(x',t)] = i\hbar\,\delta(x-x')$$

Recall "Pictures"

1) Schroedinger picture

Operator $\hat{q}_s, \hat{p}_s, \hat{O}_s$ time independent

States $|\Psi(t)\rangle = e^{-i\frac{Ht}{\hbar}}|\Psi(0)\rangle$ time dependent

2) Heisenberg picture

- States are time independent $|\Psi_H\rangle = e^{+i\frac{Ht}{\hbar}}|\Psi_s(t)\rangle$
- operator time dependent $\hat{O}_H(t) = e^{i\frac{Ht}{\hbar}}\,\hat{O}_s\,e^{-i\frac{Ht}{\hbar}}$

Observables same

3) Interactions picture (review)

$$H = H_0 + H_I$$

$$|\psi_I(t)\rangle = e^{+iH_0 t/\hbar} \, |\psi_s(t)\rangle$$

$$\hat{O}_I = e^{iH_0 t/\hbar} \, \hat{O}_s \, e^{-iH_0 t/\hbar}$$

$$|\psi_I(t)\rangle = U_I(t, t_0) \, |\psi_I(t_0)\rangle$$

$\leftarrow H_I$

Implicitly in Heisenberg picture

$$y_1(t) \rightarrow \phi(x, t)$$

Natural in field theory

$$|n\rangle = a^\dagger(p)\,|0\rangle$$

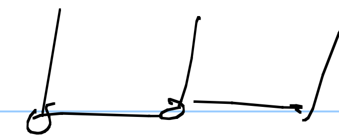
Commutation rule are equal time

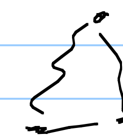
$$[\hat{q}_i(t), \hat{p}_j(t)] = [e^{\frac{iHt}{\hbar}} \hat{q}_i e^{-\frac{iHt}{\hbar}}, e^{\frac{iHt}{\hbar}} \hat{p}_j e^{-\frac{iHt}{\hbar}}]$$

$$= e^{+\frac{iHt}{\hbar}} [\hat{q}_i, \hat{p}_j] e^{-iHt/\hbar} = i\hbar \delta_{ij}$$

$$[\hat{q}_i(t), \hat{p}_j(t')] \neq i\hbar \delta_{ij} \qquad t \neq t'$$

Another view — Normal modes





$$L = \sum_i \left[m \dot{q}_i^2 + V(q) \right]$$

$$V = \frac{1}{2} N_{ij}\, q_i\, q_j \quad \Rightarrow \quad \det(m - \omega^2 N) = 0$$

$$q_i = A_{is}\, \xi_s$$

$\uparrow$ modal matrix

$$L = \sum_s \left(\frac{1}{2} \dot{\xi}_s^2 - \frac{1}{2} \omega_s^2 \xi_s^2 \right)$$

$$p_s = \frac{\partial L}{\partial \dot{\xi}_s} \qquad [\xi_s, p_{s'}] = i\hbar\, \delta_{ss'}$$

Lesson $\Rightarrow$ solve for normal modes + quantize

Field Theory normal modes = plane waves $e^{ik\cdot x}$ = "∞ # of H.O."

Continuum + 3D

$$S = \int dt\, d^3x \frac{1}{2}\left[\frac{1}{v^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - (\vec{\nabla}\phi)^2\right]$$

Label states $\vec{p} = \hbar \vec{k}$

$$\phi(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3} N_p \left[e^{-i(\omega t - \vec{k}\cdot x)} a(\vec{p}) + e^{+i(\omega t - \vec{k}\cdot\vec{x})} a^\dagger(p) \right]$$

$$e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$$

Norm

$$[a(p), a^\dagger(p')] = (2\pi\hbar)^3 \delta^3(p - p')$$

$$N_p = \sqrt{\frac{\hbar v^2}{2\omega_p}} \longleftarrow [\phi, \pi] = i\hbar\, \delta^3(x - x')$$

$$H = \int \frac{d^3p}{(2\pi\hbar)^3} \hbar\omega \frac{1}{2}\left[a^\dagger(p)a(p) + a(p)a^\dagger(p)\right] = H_0 + E_0$$

$$H_0 = \int \frac{d^3p}{(2\pi\hbar)^3} \hbar\omega \; a^\dagger(p)a(p) \qquad \star\star$$

$$E_0 = \int \frac{d^3p}{(2\pi\hbar)^3} \hbar\omega \frac{1}{2} \underbrace{(2\pi\hbar)^3 \delta^3(0)}_{\text{Volume}}$$

zero pt energy

$(2\pi)^3 \delta^3(p) = \int d^3x \, e^{ip\cdot x}$

E_0/V also divergent

State

$$|\vec{p}_1\rangle = a^\dagger(\vec{p}_1)|0\rangle \qquad a(\vec{p})|0\rangle = 0 \quad \text{all } \vec{p}$$

$$H_0|\vec{p}_1\rangle = \int \frac{d^3p}{(2\pi\hbar)^3}\, \hbar\omega_p\, a^\dagger(p)\, \underbrace{a(p)\, a^\dagger(p_1)}_{\left(\underbrace{[a(p), a^\dagger(p_1)]}_{(2\pi\hbar)^3 \delta^3(p-p_1)} + a^\dagger(p_1)\cancel{a(p)}\right)}|0\rangle$$

$$= \hbar\omega_1 |\vec{p}_1\rangle \quad \checkmark$$

Norm

$$\langle p_1|p_2\rangle = \langle 0|\, a(p_1)\, a^\dagger(p_2)|0\rangle = (2\pi\hbar)^3 \delta^3(\vec{p}_1 - \vec{p}_2)$$

Quantum Field

"Canonical quantization" $\Rightarrow$ field becomes operator

path integral quantization $\Rightarrow$ field just a field (number(x,t))

Field operator 2 parts

- wavefunctions $\leftarrow$ dynamics
- creation annih operators $\leftarrow$ book keeping $\leftarrow$ disappear in matrix elements

$$\phi(x,t) = \int d^3p \left[e^{i(Et - \vec{p}\cdot x)/\hbar} a(p) + \ldots h.c. \right]$$

wavefunctions

In cavity

$$\psi(x,t) = \sum_n N_n \left[\psi_n(x,t)\, a_n \right]$$

ψ_n ↑ wavefunctions; a_n ↑ creation op

Matrix elements

$$\langle 0 | \phi(x,t) | p_1 \rangle = N_p \, e^{-i(E_1 t - \vec{p}_1 \cdot \vec{x})/\hbar}$$

N_p ↑ $\sqrt{\dfrac{\hbar N^2}{2\omega_1}}$

$$H = V(x)\, \psi^*(x)\, \psi(x)$$

Transition

$$\langle p_2 | \phi^2(x) | p_1 \rangle = N_{p_1} N_{p_2}\, e^{-i(E_1 t - p_1 \cdot x)/\hbar}\, e^{+i(E_2 t - \vec{p}_2 \cdot \vec{x})}$$