Appendix B
Advanced field theoretic methods

B–1 The heat kernel

When using path integral techniques one must often evaluate quantities of the form

$$H(x,\tau) \equiv \left\langle x \left| e^{-\tau\mathcal{D}} \right| x \right\rangle \quad , \tag{1.1}$$

where $\mathcal{D}$ is a differential operator and τ is a parameter. In this section, we shall describe the *heat kernel* method by which $H(x,\tau)$ is expressed as a power series in τ. For example, if in d dimensions the differential operator $\mathcal{D}$ is of the form

$$\mathcal{D} = \Box + m^2 + V \ , \tag{1.2}$$

where V is some interaction, then the heat kernel expansion for $H(x,\tau)$ is

$$H(x,\tau) = \frac{i}{(4\pi)^{d/2}} \, \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \dots \right] \ . \tag{1.3}$$

where $a_i(x)$ are coefficients which will be determined below.

Let us begin by citing the two most common occurrences of $H(x,\tau)$. One is in the evaluation of the functional determinant

$$\det \mathcal{D} = e^{\mathrm{tr}\,\ln\mathcal{D}} = e^{\int d^4x \ \ \mathrm{Tr}\,\langle x|\ln\mathcal{D}|x\rangle} \quad , \tag{1.4}$$

where ‘Tr’ is a trace over internal variables like isospin, Dirac matrices, *etc.*, and ‘tr’ is a trace over these plus spacetime. The (generally singular) matrix element $\langle x|\ln\mathcal{D}|x\rangle$ appearing in Eq. (1.4) can be expressed in a variety of ways. For example, in dimensional regularization one can use the identity

$$\ln\frac{b}{a} = \int_0^\infty \frac{dx}{x} \ \left(e^{-ax} - e^{-bx} \right) \tag{1.5}$$

to write

$$\langle x|\ln\mathcal{D}|x\rangle = -\int_0^\infty \frac{d\tau}{\tau}\left\langle x\left|e^{-\tau\mathcal{D}}\right|x\right\rangle + C\ , \tag{1.6}$$

where C is a divergent constant having no physical consequences. Substituting Eq. (1.3) into the above yields

$$\langle x|\ln\mathcal{D}|x\rangle - C = -\frac{i}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} m^{d-2n}\ \Gamma\left(n-\frac{d}{2}\right)a_n(x)\ \ . \tag{1.7}$$

The divergences in the series representation arise from the Γ-function and are restricted in four dimensions to the terms $a_0(x), a_1(x), a_2(x)$.

The heat kernel can likewise be used to analyze the functional determinant in alternative regularization procedures, such as zeta-function regularization. Here, one expresses the matrix element $\langle x|\ln\mathcal{D}|x\rangle$ as

$$\begin{aligned}\langle x|\ln\mathcal{D}|x\rangle &= -\left\langle x\left|\left[\frac{d}{ds}e^{-s\ln\mathcal{D}}\right]_{s=0}\right|x\right\rangle \\ &= -\left[\frac{d}{ds}\left\langle x\left|\frac{1}{\mathcal{D}^s}\right|x\right\rangle\right]_{s=0} = -\frac{d}{ds}\xi_{\mathcal{D}}(x,s)\Big|_{s=0}\ , \\ \xi_{\mathcal{D}}(x,s) &\equiv \frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}H(x,\tau)\ \ .\end{aligned} \tag{1.8}$$

The penultimate equality in Eq. (1.8) is obtained from repeated formal differentiation of Eq. (1.6) with respect to $\mathcal{D}$. Upon expanding the $H(x,\tau)$ term in $\xi_{\mathcal{D}}(x,s)$, one arrives at the desired power series expansion of $\langle x|\ln\mathcal{D}|x\rangle$. This usage is applied in the next section.

The other main use of the heat kernel is in the regularization of anomalies. Often one is faced with making sense of Tr $\langle x\,|O(x)|\,x\rangle$, where O is a local operator. Although such quantities are generally singular, they can be defined in a gauge-invariant manner by damping out the contributions of large eigenvalues,

$$\text{Tr}\ \langle x\,|O(x)|\,x\rangle = \lim_{\epsilon\to 0}\ \text{Tr}\ \left\langle x\left|O(x)e^{-\epsilon\mathcal{D}}\right|x\right\rangle\ , \tag{1.9}$$

where $\mathcal{D}$ is a gauge-invariant differential operator. Again it is only the low-order coefficients, generally those up to $a_2(x)$, which contribute in the $\epsilon\to 0$ limit. We employ this technique in Sects. III–3,4.

As an example of heat kernel techniques, let us consider the following operator defined in d dimensions:

$$\mathcal{D} = d_\mu d^\mu + m^2 + \sigma(x) \qquad (d_\mu \equiv \frac{\partial}{\partial x^\mu} + \Gamma_\mu(x))\ , \tag{1.10}$$

where $\Gamma_\mu(x)$ and $\sigma(x)$ are functions and/or matrices defined in some internal symmetry space. In particular, neither Γ_μ nor σ contains derivative operators. Employing a complete set of momentum eigenstates $\{|p\rangle\}$

allows us to express the heat kernel as

$$H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\ e^{-ip\cdot x} e^{-\tau\mathcal{D}} e^{ip\cdot x} \quad , \tag{1.11}$$

where in d dimensions, use is made of the relations

$$\begin{aligned} \langle p|x\rangle &= \frac{1}{(2\pi)^{d/2}} e^{ip\cdot x} \ , \\ \langle x|x'\rangle &= \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x'-x)} = \delta^{(d)}(x-x') \ , \\ \langle p'|p\rangle &= \int \frac{d^d x}{(2\pi)^d} e^{i(p'-p)\cdot x} = \delta^{(d)}(p'-p) \quad . \end{aligned} \tag{1.12}$$

From the identities

$$\begin{aligned} d_\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu) \ , \\ d_\mu d^\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu)(ip^\mu + d^\mu) \ , \end{aligned} \tag{1.13}$$

we can then write

$$\begin{aligned} H(x,\tau) &= \int \frac{d^d p}{(2\pi)^d}\ e^{-\tau\left[(ip_\mu + d_\mu)^2 + m^2 + \sigma\right]} \\ &= \int \frac{d^d p}{(2\pi)^d}\ e^{\tau[p^2 - m^2]} e^{-\tau[d\cdot d + \sigma + 2ip\cdot d]} \quad . \end{aligned} \tag{1.14}$$

The first exponential factor is simply the free field result, while all the interesting physics is in the second exponential. The latter can be Taylor expanded in powers of τ, keeping those terms which contribute up to order τ^2 after integration over momentum. Note that each power of p^2 contributes a factor of $1/\tau$. Thus we obtain the expansion

$$\begin{aligned} H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\ e^{\tau(p^2 - m^2)} \Bigg[& 1 - \tau[d\cdot d + \sigma] \\ & + \frac{\tau^2}{2}\left[(d\cdot d+\sigma)(d\cdot d+\sigma) - 4p\cdot d\ p\cdot d\right] \\ & + \frac{4\tau^3}{3!}[p\cdot d\ p\cdot d(d\cdot d+\sigma) + p\cdot d(d\cdot d+\sigma)p\cdot d \\ & \quad + (d\cdot d+\sigma)p\cdot d\ p\cdot d] \\ & + \frac{16\tau^4}{4!} p\cdot d\ p\cdot d\ p\cdot d\ p\cdot d + \ldots \Bigg] \ , \end{aligned} \tag{1.15}$$

where terms odd in p have been dropped and we have displayed only those $\mathcal{O}(\tau^3)$ and $\mathcal{O}(\tau^4)$ terms which contribute to H at order τ^2 after p is integrated over. To perform the integral, it is convenient to continue to euclidean momentum $p_E = \{p_1, p_2, p_3, p_4 = -ip_0\}$. Then with the replacement $p_\mu p^\mu \to -|p_E^\mu p_E^\mu| = -p_E^2$, we obtain

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} = \int \frac{d\Omega_d}{(2\pi)^d} \int dp_E\ p_E^{d-1} e^{-(p_E^2+m^2)\tau}$$
$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \frac{e^{-m^2\tau}\Gamma(d/2)}{2\tau^{d/2}}$$
$$= \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2}} \ ,$$
$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} p_E^\mu p_E^\nu = \frac{\delta^{\mu\nu}}{d} \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2+1}} \frac{\Gamma(d/2+1)}{\Gamma(d/2)} \tag{1.16}$$
$$= \frac{\delta^{\mu\nu}}{2} \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+1}} \ ,$$
$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} p_E^\mu p_E^\nu p_E^\lambda p_E^\sigma = \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+2}}$$
$$\times \frac{\left(\delta^{\mu\nu}\delta^{\lambda\sigma} + \delta^{\mu\lambda}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\lambda\nu}\right)}{4} \ .$$

Employing these relations to evaluate Eq. (1.14) gives (to second order in τ),

$$H(x,\tau) = \frac{ie^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2}}$$
$$\times \left[1 - \tau\sigma + \tau^2 \left(\frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu,[d^\mu,\sigma]]\right)\right] \ , \tag{1.17}$$

or in the notation of Eq. (1.3),

$$a_0(x) = 1 \ , \qquad a_1(x) = -\sigma \ ,$$
$$a_2(x) = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}\,[d_\mu,[d^\mu,\sigma]] \ . \tag{1.18}$$

Fermions are treated in a similar manner. For example, the identity

$$\ln \not{D} = \frac{1}{2}\ln(\not{D}\,\not{D}) \tag{1.19}$$

allows the same technique to be used for the operator $\not{D}\,\not{D}$. In particular let us consider the case where

$$\not{D} = \not{\partial} + i\,\not{V} + i\,\not{A}\gamma_5 \ . \tag{1.20}$$

With some work, one can cast this into the form of Eq. (1.9) with the identifications

$$\begin{aligned} \not{D}\,\not{D} &\equiv \mathcal{D} = d_\mu d^\mu + \sigma \ , \\ d_\mu &= \partial_\mu + iV_\mu + \sigma_{\mu\nu}A^\nu\gamma_5 \equiv \partial_\mu + \Gamma_\mu \ , \\ \sigma &= \frac{1}{2}\sigma_{\mu\nu}V^{\mu\nu} - 2A_\mu A^\mu + (i\partial_\mu A^\mu - [V_\mu, A^\mu])\,\gamma_5 \ , \\ V_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] + i[A_\mu, A_\nu] \quad . \end{aligned} \tag{1.21}$$

The values of $a_i(x)$ appearing in Eq. (1.18) can also be used in this case. The heat kernel coefficients have been worked out for more general situations [Gi 75].

B–2 Chiral renormalization and background fields

In this section, we illustrate the method described above while also proving an important result for the theory of chiral symmetry. The goal is to demonstrate that all the divergences encountered at one loop can be absorbed into a renormalization of the coefficients of the $\mathcal{O}(E^4)$ chiral lagrangian and to identify the renormalization constants. The technique used here, the *background field method*, is of considerable interest in its own right [Sc 51, De 67, Ab 82, Bal 89] and is applicable to areas such as general relativity [BiD 82].

The basic idea of the background field method is to calculate quantum corrections about some nonvanishing field configuration $\overline{\varphi}$,

$$\varphi(x) = \bar{\varphi}(x) + \delta\varphi(x) \quad , \tag{2.1}$$

rather than about the zero field,* and to then compute the path integral over the fluctuation $\delta\varphi(x)$. The result is an effective action for $\bar{\varphi}$. This effective action can be expanded in powers of $\overline{\varphi}$ and applied to matrix elements at tree level, resulting in a description of scattering processes at one-loop order. In the case of the chiral lagrangian, one expands the full chiral matrix

$$U = \bar{U} + \delta U \quad , \tag{2.2}$$

where $\bar{U}$ satisfies the classical equation of motion. Upon integration over δU, one obtains the one-loop effective action for $\bar{U}$. This contains a great deal of information. In particular, $\bar{U}$ can be expanded in the usual way in terms of a set of external meson fields

$$\bar{U} = \exp(i\lambda^a\bar{\varphi}^a/F) \quad (a = 1, \ldots, 8) \quad . \tag{2.3}$$

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* See the discussion in Appendix A–4.

Contained in $S_{\rm eff}(\bar{U})$ is the effective one-loop action for arbitrary numbers of meson fields. Upon identification of renormalization constants, all processes become renormalized at the same time.

Our starting point is, in the notation of Sect. IV–6, the $\mathcal{O}(E^2)$ lagrangian

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,{\rm Tr}\,\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_0^2}{4}\,{\rm Tr}\,\left(\chi^\dagger U + U^\dagger \chi\right) \quad . \tag{2.4}$$

The procedure to follow is rather technical, so let us first quote the end result of the calculation. Upon performing the one-loop quantum corrections, the effective action will have the form

$$S_{\rm eff} = S_2^{\rm ren} + S_4^{\rm ren} + S_4^{\rm finite} + \ldots \quad .$$

Here the lagrangians in $S_2^{\rm ren}$, $S_4^{\rm ren}$ are the ones quoted in Sect. VII–2, but now with renormalized coefficients. In particular $S_4^{\rm ren}$ is the sum $S_4^{\rm ren} = S_4^{\rm bare} + S_4^{\rm div}$ where, in chiral $SU(3)$ and employing dimensional regularization, $S_4^{\rm div}$ is given by

$$\begin{aligned} S_4^{\rm div} = -\lambda \int d^4x \Bigg[& \frac{3}{32}\left[{\rm Tr}\,\left(D_\mu U D^\mu U^\dagger\right)\right]^2 \\ & + \frac{3}{16}\,{\rm Tr}\,\left(D_\mu U D_\nu U^\dagger\right)\,{\rm Tr}\,\left(D^\mu U D^\nu U^\dagger\right) \\ & + \frac{1}{8}\,{\rm Tr}\,\left(D_\mu U D^\mu U^\dagger\right)\,{\rm Tr}\,\left(\chi^\dagger U + U^\dagger \chi\right) \\ & + \frac{3}{8}\,{\rm Tr}\,\left[D_\mu U D^\mu U^\dagger\left(\chi U^\dagger + U\chi^\dagger\right)\right] \\ & + \frac{11}{144}\left[{\rm Tr}\,\left(\chi U^\dagger + U\chi^\dagger\right)\right]^2 + \frac{5}{48}\,{\rm Tr}\,\left(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger\right) \\ & + \frac{i}{4}\,{\rm Tr}\,\left(L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U^\dagger D^\nu U\right) - \frac{1}{4}\,{\rm Tr}\,\left(L_{\mu\nu} U R^{\mu\nu} U^\dagger\right)\Bigg] \end{aligned} \tag{2.5}$$

with

$$\lambda \equiv \frac{1}{32\pi^2}\left\{\frac{2}{d-4} - \ln 4\pi - 1 + \gamma\right\} \quad . \tag{2.6}$$

The terms in $S_4^{\rm div}$ are all of the same form as the terms in the bare lagrangian at order E^4. Therefore, all the divergences can be absorbed into renormalized values of these constants. The finite remainder, $S_4^{\rm finite}$, cannot be simply expressed as a local lagrangian, but can be worked out for any given transition. When $S_4^{\rm div}$ is added to the $\mathcal{O}(E^4)$ tree-level lagrangian of Eq. (VII–2.7), the result has the same form but with coefficients

$$\alpha_i^r = \alpha_i - \gamma_i \lambda \quad , \tag{2.7}$$

where the $\{\gamma_i\}$ are numbers which are given in Table B–1 for both the case of chiral $SU(2)$ and $SU(3)$. Thus the divergences can all be absorbed into the redefined parameters and these in turn can be determined from experiment. Let us now turn to the task of obtaining this result.

In applying the background field method, there are a variety of ways to parameterize δU, and several different ones are used in the literature. The prime consideration is to maintain the unitarity property $U^\dagger U = 1 = \left(\bar{U}^\dagger + \delta U^\dagger\right)\left(\bar{U} + \delta U\right)$ along with $\bar{U}^\dagger \bar{U} = 1$. We shall take

$$U = \bar{U} e^{i\Delta} \ , \tag{2.8}$$

with $\Delta \equiv \lambda^a \Delta^a$ representing the quantum fluctuations. This choice is made to simplify the algebra in the heat kernel renormalization approach, which we shall describe shortly. Another possible choice is

$$U = \xi e^{i\eta} \xi \tag{2.9}$$

with $\eta = \lambda^a \eta^a$ and $\xi\xi \equiv \bar{U}$. These two forms are related by $\eta = \xi \Delta \xi^\dagger$. Since in the path integral, we integrate over all values of Δ (or η) at each point of spacetime, these two choices are equivalent.

The expansion of the lagrangian in terms of $\bar{U}$ and Δ is straightforward, and we find

$$\begin{aligned}
\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) &= \mathrm{Tr}\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right) - 2i\,\mathrm{Tr}\left(\bar{U}^\dagger D_\mu \bar{U} \tilde{D}^\mu \Delta\right) \\
&+ \mathrm{Tr}\left[\tilde{D}_\mu \Delta \tilde{D}^\mu \Delta + \bar{U}^\dagger D_\mu \bar{U}\left(\Delta \tilde{D}^\mu \Delta - \tilde{D}^\mu \Delta\, \Delta\right)\right] \ , \\
\mathrm{Tr}\left(\chi^\dagger U + U^\dagger \chi\right) &= \mathrm{Tr}\left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi\right) + i\,\mathrm{Tr}\left(\Delta\left(\chi^\dagger \bar{U} - \bar{U}^\dagger \chi\right)\right) \\
&- \frac{1}{2}\,\mathrm{Tr}\left[\Delta^2\left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi\right)\right] \ ,
\end{aligned} \tag{2.10}$$

where

$$\tilde{D}_\mu \Delta \equiv \partial_\mu \Delta + i\left[r_\mu, \Delta\right] \ . \tag{2.11}$$

Table B–1. Renormalization coefficients

	i	1	2	3	4	5	6	7	8	9	10
$SU(2)$	γ_i	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{32}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$
$SU(3)$	γ_i	$\frac{3}{32}$	$\frac{3}{16}$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{11}{144}$	0	$\frac{5}{48}$	$\frac{1}{4}$	$-\frac{1}{4}$

Since $\bar{U}$ satisfies the equation of motion, there is no term linear in Δ. One may integrate various terms in the action by parts to obtain

$$S_2^{(0)} = \int d^4x \left\{ \mathcal{L}_2(\bar{U}) - \frac{F_0^2}{2} \Delta^a \left(d_\mu d^\mu + \sigma \right)^{ab} \Delta^b + \ldots \right\} \tag{2.12}$$

where

$$\begin{aligned} d_\mu^{ab} &= \delta^{ab}\partial_\mu + \Gamma_\mu^{ab} \;, \\ \Gamma_\mu^{ab} &= -\frac{1}{4} \,\mathrm{Tr}\, \left(\left[\lambda^a, \lambda^b \right] \left(\bar{U}^\dagger \partial_\mu \bar{U} + i\bar{U}^\dagger \ell_\mu \bar{U} + ir_\mu \right) \right) \;, \\ \sigma^{ab} &= \frac{1}{8} \,\mathrm{Tr}\, \left(\left\{ \lambda^a, \lambda^b \right\} \left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right) + \left[\lambda^a, \bar{U}^\dagger D_\mu \bar{U} \right] \left[\lambda^b, \bar{U}^\dagger D^\mu \bar{U} \right] \right) \;. \end{aligned} \tag{2.13}$$

The action is now a simple quadratic form, and the path integral may be performed. The only potential complication is the question of interpreting the integration variables. This is referred to as the 'question of the path integral measure'. The integration over all the unitary matrices U can be accomplished by an integration over the parameters in the exponential

$$\int [dU] = N \int [d\Delta^a] \;, \tag{2.14}$$

where N is a constant which plays no dynamical role. With this identification one obtains

$$\begin{aligned} e^{iW_{\mathrm{loop}}} &= \int [d\Delta^a] e^{i\int d^4x \frac{F^2}{2} \Delta^a (d_\mu d^\mu + \sigma)^{ab} \Delta^b} \\ &= \left(\det \left[d_\mu d^\mu + \sigma \right] \right)^{-1/2} = \exp \left\{ -\frac{1}{2} \,\mathrm{tr}\, \ln \left(d_\mu d^\mu + \sigma \right) \right\} \;. \end{aligned} \tag{2.15}$$

Here ' tr ' indicates a trace over the spacetime indices as well as over the $SU(N)$ indices a, b.

The identification of divergences is most conveniently done by using the heat kernel expansion derived earlier in this appendix, where it is shown that all the ultraviolet divergences are contained in the first few expansion coefficients. The relevant terms are

$$\begin{aligned} W_{\mathrm{loop}} &= \frac{i}{2} \,\mathrm{tr}\, \ln \left(d_\mu d^\mu + \sigma \right) \\ &= \frac{1}{2(4\pi)^{d/2}} \int d^4x \lim_{m \to 0} \left\{ \Gamma \left(1 - \frac{d}{2} \right) m^{d-2} \,\mathrm{Tr}\, \sigma \right. \\ &\left. + m^{d-4} \Gamma \left(2 - \frac{d}{2} \right) \,\mathrm{Tr}\, \left(\frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2 \right) + \ldots \right\} \;, \end{aligned} \tag{2.16}$$

where

$$\Gamma_{\mu\nu}^{ab} = \partial_\mu \Gamma_\nu^{ab} - \partial_\nu \Gamma_\mu^{ab} + \Gamma_\mu^{ac} \Gamma_\nu^{cb} - \Gamma_\nu^{ac} \Gamma_\mu^{cb} = \left[d_\mu, d_\nu \right]^{ab} \;. \tag{2.17}$$

For N_f flavors, the operator part of the first term in Eq. (2.16) is

$$\mathrm{Tr}\,\sigma = \frac{N_f}{2}\,\mathrm{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right) + \frac{N_f^2-1}{2N_f}\,\mathrm{Tr}\,\left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi\right) \ . \tag{2.18}$$

The above two traces are just those which appear in $\mathcal{L}_2^{(0)}$, so that they can only modify F_π and m_π^2. The remaining terms can be worked out with a bit more algebra. Using the identity

$$\partial_\mu\left(\bar{U}^\dagger \partial_\nu \bar{U}\right) - \partial_\nu\left(\bar{U}^\dagger \partial_\mu \bar{U}\right) = -\left[\bar{U}^\dagger \partial_\mu U, \bar{U}^\dagger \partial_\nu U\right] \ , \tag{2.19}$$

we find for the field strength,

$$\Gamma_{\mu\nu}^{ab} = \frac{1}{8}\,\mathrm{Tr}\,\left\{\left[\lambda^a, \lambda^b\right]\left(\left[\bar{U}^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U}\right] + i\bar{U}^\dagger L_{\mu\nu}\bar{U} + iR_{\mu\nu}\right)\right\} \ . \tag{2.20}$$

This produces, for N_f flavors in chiral $SU(N_f)$,

$$\begin{aligned}
\mathrm{Tr}\,(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) = {} & \frac{N_f}{8}\,\mathrm{Tr}\,\left(\left[\bar{U}^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U}\right]\left[\bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}\right]\right) \\
& + iN_f\,\mathrm{Tr}\,\left(R_{\mu\nu}\partial^\mu \bar{U}^\dagger \partial^\nu \bar{U} + L_{\mu\nu}\partial^\mu \bar{U}\partial^\nu \bar{U}^\dagger\right) \\
& - N_f\,\mathrm{Tr}\,\left(L_{\mu\nu}\bar{U}R^{\mu\nu}\bar{U}^\dagger\right) - \frac{N_f}{2}\,\mathrm{Tr}\,\left(L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\right) \ , \\
\mathrm{Tr}\,\sigma^2 = {} & \frac{1}{8}\left[\mathrm{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right)\right]^2 + \frac{1}{4}\,\mathrm{Tr}\,\left(D_\mu \bar{U} D_\nu \bar{U}^\dagger\right)\,\mathrm{Tr}\,\left(D^\mu \bar{U} D^\nu \bar{U}^\dagger\right) \\
& + \frac{N_f}{8}\,\mathrm{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger D_\nu \bar{U} D^\nu \bar{U}^\dagger\right) + \frac{2+N_f^2}{8N_f^2}\left[\mathrm{Tr}\,\left(\chi \bar{U}^\dagger + \bar{U}^\dagger \chi\right)\right]^2 \\
& + \frac{1}{4}\,\mathrm{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right)\,\mathrm{Tr}\,\left(\chi \bar{U}^\dagger + \bar{U}\chi^\dagger\right) \\
& + \frac{N_f}{4}\,\mathrm{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\left(\chi \bar{U}^\dagger + \bar{U}\chi^\dagger\right)\right) \\
& + \frac{N_f^2-4}{8N_f}\,\mathrm{Tr}\,\left(\left(\chi \bar{U}^\dagger + \bar{U}\chi^\dagger\right)\left(\chi \bar{U}^\dagger + \bar{U}\chi^\dagger\right)\right) \ .
\end{aligned} \tag{2.21}$$

The only operator which is not of the same form as the basic $\mathcal{O}(E^4)$ lagrangian occurs in the first term of $\mathrm{Tr}\,\Gamma^2$. However, by use of Eq. (VII–2.3) for $SU(3)$, it can be written as a linear combination of our standard forms. For $N_f = 3$, these add up to the result previously quoted in Eq. (2.5). Here the divergence is in the parameter λ. For convenience in applications, we have added some finite terms to the definitions of λ. The results for $N_f = 2$ are also quoted in Table B–1, although some of the operators are redundant for that case.

The reader who has understood the above development as well as the standard perturbative methods presented in the main text will be prepared for the use of the background field method in the full calculation of transition amplitudes. This procedure consists of writing

$$\begin{aligned} & d_\mu d^\mu + \sigma = \mathcal{D}_0 + V \\ & \mathcal{D}_0 = \Box + m^2 \\ & V = \{\partial_\mu, \Gamma^\mu\} + \Gamma_\mu \Gamma^\mu + \sigma - m^2 \quad , \end{aligned} \tag{2.22}$$

where m^2 is the meson mass-squared matrix. The one-loop action is then expanded in powers of the interaction V

$$\begin{aligned} W_{loop} &= \frac{i}{2} \text{tr} \ln(d_\mu d^\mu + \sigma) = \frac{i}{2} \text{tr} \left[\ln \mathcal{D}_0 + \ln(1 + \mathcal{D}_0^{-1} V) \right] \\ &= \frac{i}{2} \text{tr} \left[\ln \mathcal{D}_0 + \mathcal{D}_0^{-1} V - \frac{1}{2} \mathcal{D}_0^{-1} V \mathcal{D}_0^{-1} V + \ldots \right] \quad . \end{aligned} \tag{2.23}$$

The first term is an uninteresting constant which may be dropped, and the remainder has the coordinate space form

$$\begin{aligned} W_{loop} = & -\frac{i}{2} \int d^4x \, \text{Tr} \, [\Delta_F(x-x) V(x)] \\ & - \frac{i}{4} \int d^4x d^4y \, \text{Tr} \, [\Delta_F(x-y) V(y) \Delta_F(y-x) V(x)] + \ldots \quad . \end{aligned} \tag{2.24}$$

When the matrix elements of this action are taken, the result contains not only the divergent terms calculated above, but also the finite components of the one-loop amplitudes. The resulting expressions are presented fully in [GaL 84,GaL 85a]. This method allows one to calculate the one-loop corrections to many processes at the same time, and in practice is a much simpler procedure for some of the more difficult calculations.

B–3 PCAC and the soft-pion theorem

We have emphasized the use of effective lagrangians to elucidate the symmetry predictions of a theory. For a dynamically broken chiral symmetry such as QCD these predictions will relate processes which have different numbers of Goldstone bosons. The machinery of effective lagrangians will correctly yield such predictions, but it is often useful to have an alternative technique for understanding or calculating these results. In the case of chiral symmetry, this is provided by the *soft-pion theorem* which explicitly relates a process with a pion to one with that pion removed from the amplitude. Calculations performed this way uses current algebra methods which go by the name of *Partial Conservation*