Week 1: Lectures 1 & 2.

Outline:

* Second quantization
* Canonical transformation
* Quasi particles
* Examples: fermionic chain
* Jordan-Wigner transformation
* 1D Heisenberg spin-$\frac{1}{2}$ chain
* XXZ chain.
* Quasiparticles: 1. Second quantization; Canonical transformation

The systems with large number of identical particles are convenient to study using the second quantization method.

a) Consider a system of boson particles, each of which can find itself in one of the following states:

\[ \Phi_1(x), \Phi_2(x), \Phi_3(x), \ldots = \Phi_i(x), i=1,2 \ldots \]

Many-body wavefunction is thus given by occupation numbers, which show how many particles occupy the given state \( \Phi_i(x) \). In Dirac’s notation, such states can be written as

\[ \ldots, N_{i-1}, N_i, N_{i+1}, \ldots \rightarrow \], where

occupation numbers \( N_i \) acquire arbitrary positive integer numbers \( N_i = 0, 1, 2, \ldots \).

Canonical creation and annihilation operators \( a_i^+, a_i \) are introduced as follows:

\[ a_i \left| \ldots N_i, N_{i+1}, \ldots \right> = \sqrt{N_i} \left| \ldots, N_{i+1}, N_{i+1}, \ldots \right> \]

\[ a_i^+ \left| \ldots N_i, N_{i+1}, \ldots \right> = \sqrt{N_i + 1} \left| \ldots, N_i + 1, N_{i+1}, \ldots \right> \]
From this definition, we can easily compute commutation relations of creation and annihilation operators:

\[ [a_i, a_j^+] = a_i a_j^+ - a_j^+ a_i = \delta_{ij} \]

**Proof:** Let \( i \neq j \Rightarrow a_i a_j^+ | \ldots N_i \ldots N_j \ldots \rangle = \]
\[ = \sqrt{N_i} \sqrt{N_j+1} | \ldots N_i-1 \ldots N_j+1 \ldots \rangle \]
and
\[ a_j^+ a_i | \ldots N_i \ldots N_j \ldots \rangle = \sqrt{N_j+1} \sqrt{N_i} \times \]
\[ \times | \ldots N_i-1 \ldots N_j+1 \ldots \rangle \]
\[ \Rightarrow (a_i a_j^+ - a_j^+ a_i) | \ldots N_i \ldots N_j \ldots \rangle = 0. \Rightarrow [a_i, a_j^+] = 0 \]

If \( i = j \),
\[ [a_i, a_i^+] = (a_i a_i^+ - a_i^+ a_i) | \ldots N_i \ldots \rangle = \]
\[ = \sqrt{N_i+1} \sqrt{N_i+1} | \ldots N_i \ldots \rangle - \]
\[ - \sqrt{N_i} \sqrt{N_i} | \ldots N_i \ldots \rangle = [N_i+1 - N_i] | \ldots N_i \ldots \rangle \]
\[ = | \ldots N_i \ldots \rangle \]
\[ \Rightarrow (a_i a_i^+ - a_i^+ a_i) = 1 \Rightarrow \text{identity operator.} \]
Similarly, one can show that

$$[a_i, a_i^+] = [a_i^+, a_i^+] = 0.$$  

As the next step, one defines a  

$$\hat{\Psi}(x) = \sum_i \hat{a}_i \psi_i(x) \quad \hat{\Psi}^+(x) = \sum_i \hat{a}_i^+ \psi_i^+(x).$$  

Functions $\psi_i(x)$, $i = 1, 2, \ldots$ are chosen to compose a full orthonormal set, i.e.,

$$\int \psi_i(x) \psi_j(x) \, dx = 0, \quad i \neq j$$

$$\| \psi_i(x) \| = \sqrt{\int |\psi_i(x)|^2 \, dx} = 1.$$  

From here, one can obtain commutation relations of $\hat{\Psi}$-operators as follows

$$[\hat{\Psi}(x), \hat{\Psi}^+(x')] = \delta(x-x')$$

$$[\hat{\Psi}(x), \hat{\Psi}(x')] = [\hat{\Psi}^+(x), \hat{\Psi}^+(x')] = 0.$$
b) Consider a system of Fermi statistics. Then the formal definitions of occupation number and creation/annihilation operators are similar. The main difference follows from Pauli's principle, which implies that $N_i = 0, 1$ only.

Therefore, canonical operators $a_i^+, a_i$ act as follows:

$$ a_i | \ldots, N_i, N_{i+1}, \ldots > = \begin{cases} | \ldots, 0, N_{i+1}, \ldots >, & \text{if } N_i = 1 \\ 0, & \text{if } N_i = 0 \end{cases} $$

$$ a_i^+ | \ldots, N_i, N_{i+1}, \ldots > = \begin{cases} 0, & \text{if } N_i = 1 \\ | 0, \ldots, 1, N_{i+1}, \ldots >, & \text{if } N_i = 0 \end{cases} $$

Secondly, the asymmetry of many-particle state with respect to exchange of particles implies anticommutativity of $a_i, a_i^+$:

$$ \{ a_i^+, a_j \} = a_i^+ a_j + a_j a_i^+ = \delta_{ij} $$

$$ \{ a_i, a_j \} = \{ a_i^+, a_i^+ \} = 0. $$

Anticommutativity of $\hat{\Psi}$-operators can be cast as follows:

$$ \{ \hat{\Psi}(x), \hat{\Psi}^+(x') \} = \delta(x-x'), \quad \{ \hat{\Psi}(x), \hat{\Psi}(x') \} = \{ \hat{\Psi}^+(x), \hat{\Psi}^+(x') \} = 0. $$
Hamiltonian of a many-body system

Second quantization establishes convenient "language" between single and many-particle systems. For example, a system of non-interacting bosons or fermions can be described using in a potential \( V(r) \), can be described by the following Hamiltonian:

\[
\hat{H}_0 = \int \left[ \frac{-\hbar^2}{2m} \nabla^2 \hat{\Psi}^+(r) \nabla \hat{\Psi}(r) + \hat{\Psi}^+(r) \hat{\Psi}(r) V(r) \right] d^3r
\]

Particles here are assumed to be non-relativistic, having a quadratic dispersion.

If these particles interact via \( V(\vec{r}_1 - \vec{r}_2) \) potentials, then the Hamiltonian has to be supplemented by

\[
\hat{H}_{\text{int}} = \frac{1}{2} \int \int \hat{\Psi}^+(\vec{r}_1) \hat{\Psi}^+(\vec{r}_2) V(\vec{r}_1 - \vec{r}_2) \hat{\Psi}(\vec{r}_1) \hat{\Psi}(\vec{r}_2) \sqrt{m_1} \sqrt{m_2} d^3r_1 d^3r_2
\]

The second-order quantized density operator

\[
\hat{\rho}(\vec{r}) = \hat{\Psi}^+(\vec{r}) \hat{\Psi}(\vec{r})
\]

is a many-body equivalent of single-particle probability density \( |\Psi(\vec{r})|^2 \).
\[ \hat{N} = \int \hat{\psi}^*(\vec{r}) \hat{\psi}(\vec{r}) \, d^3r \] is the particle number operator. So far, in this section, we'll agree (of note, hint, \( \hat{P}(\vec{r}), \hat{N} \)) all the expressions are the same for bosons and fermions.

**Canonical transformations in second quantization**

Recall that in classical mechanics, canonical transformations of phase space \((p,q) \rightarrow (p',q')\) are introduced via Poisson brackets, in such a way that any preserved Hamilton's equations of motion:

\[ \dot{p} = \{ p, H \}, \quad \dot{q} = \{ q, H \} \]

In quantum mechanics, one promotes Poisson bracket to commutator (for example in Heisenberg's scheme). Equations of motion acquire the form \( i\hbar \frac{d}{dt} \hat{A} = [\hat{A}, \hat{H}] \).

So canonical transformations by definition preserve commutation relations of operators.

Similarly to classical mechanics, canonical
transformations are important as they preserve the form of equations of motion.

By choosing a canonical transformation, one may transition from interacting particles to non-interacting quasi-particles.

Often, one considers linear canonical transformations (of bosons or fermions):

\[ \overline{a}_i = \sum_j (U_{ij} a_j + V_{ij} a_j^+ ) \] (X)

\[ \overline{a}_i^+ = \sum_j (V_{ij}^* a_j + U_{ij}^* a_j^+ ) \]

-called Bogoliubov transformation.

One of the typical problems in many-body theory is to find the spectrum of eigenstates of the Hamiltonian, if the it is quadratic in particle creation and annihilation operators. It is easy to check that such Hamiltonians can be diagonalized via Bogoliubov's transformation, s.t.

\[ \hat{H} = \sum_{ij} (1) h_{ij} \overline{a}_i^+ \overline{a}_j + (2) \overline{h}_{ij} a_i a_j + h.c. = \]

\[ = \sum_i \epsilon_i \overline{a}_i^+ \overline{a}_i + \langle 0 | \hat{H} | 0 \rangle. \]

quasiparticle spectrum: zero point energy.
Transformation \((X)\) is canonical, if it preserves commutation relations:

\[ [a_i, a_j] = 0, \quad [a_i, a^+_j] = \delta_{ij} \quad \text{for bosons} \]

\[ [a_i, a^+_j] = 0, \quad [a_i, a^+_j] = \delta_{ij} \quad \text{for fermions}. \]

or, we can write:

\[ [a_i, a_j] = 0, \quad [a_i, a^+_j] = \delta_{ij}. \]

\[ \text{Dirac brackets} \quad \text{These conditions imply that} \]

\[ U^* \text{ and } V \text{ matrices should obey:} \]

\[ U_{ki} V^*_{kj} + V_{ki} U_{kj} = 0 \]

\[ U^*_{ki} U_{ki} + V^*_{ki} V_{ki} = \delta_{ij}, \]

where \( + \) sign corresponds to fermions,

\( = \) sign corresponds to bosons.
Canonical transformation (X) for the case of single boson (e.g., quantum-mechanical oscillator).

In this case, \( U \) and \( V \) are just numbers (and not matrices). Then there are only 2 types of homogeneous canonical transformations:

\[
\begin{align*}
\tilde{a} &= e^{i\varphi} a \\
\tilde{a}^+ &= e^{-i\varphi} a
\end{align*}
\]

where \( \varphi \) is a real parameter.

More general transformations are just compositions (direct products) of these transformations.

In case of fermi-statistics, canonical transformations are defined in a similar way. For one single fermion degree of freedom, all canonical transformations amount to a taking a conjugate and multiplication by a phase

\[
\tilde{a} = e^{i\varphi} a, \quad \tilde{a}^+ = e^{-i\varphi} a
\]

More general transformations in this case are nonlinear.
More general linear transformations appear in a system of 2 fermions:

\[ \mathbf{a} = \cos \theta \mathbf{a} - \sin \theta \mathbf{b}, \quad \mathbf{a}^+ = \cos \theta \mathbf{a}^+ - \sin \theta \mathbf{b} \]
\[ \mathbf{b}^+ = \sin \theta \mathbf{a} + \cos \theta \mathbf{b}^+, \quad \mathbf{b} = \sin \theta \mathbf{a}^+ + \cos \theta \mathbf{b} \]

where \( \theta \) is a parameter.

Interestingly, the first canonical transformation for bosons is "pseudo"-Euclidean rotation, i.e., Lorentz transformation with rapidity \( \lambda \) in 2D space-time.

For 2 fermions, however, one obtains rotation of Euclidean space.

Canonical transformations are not only linear and homogeneous. Sometimes nonlinear or inhomogeneous transformations are quite useful.

Fourier transformation is a type of linear canonical transformation. For \( \Psi \) operator one has

\[ \hat{\Psi}(\mathbf{r}) = \int e^{i \mathbf{p} \cdot \mathbf{r}} \frac{d^3 \mathbf{p}}{(2\pi)^3}, \quad \hat{\Psi}^+(\mathbf{r}) = \int e^{-i \mathbf{p} \cdot \mathbf{r}} \frac{d^3 \mathbf{p}}{(2\pi)^3} \]
One normalizes $\hat{a}_p$, $\hat{a}_p^*$ in such a way that

$$[\hat{a}_p, \hat{a}_p^*]_+ = (2\pi)^d \delta^{(d)}(\vec{p}_1 - \vec{p}_2).$$

An important class of problems in solid state physics is given by $\hat{\Phi}$-operators defined on lattice sites. The Fourier transform of $\hat{\Phi}$ can be defined in a way that $\hat{\Phi}$ belongs to lattice sites and $\vec{p}$ lives in a Brillouin zone.

In 1D: $\vec{p} = a\vec{p}$, and $-\pi/a < \vec{p} < \pi/a$ is the Brillouin zone.

**Example:** Fermionic chain is given by

$$\hat{H} = \sum_{i=0}^{\infty} \left( J_1 \hat{c}_i^+ \hat{c}_{i+1} + J_1 \hat{c}_{i+1}^+ \hat{c}_i + J_2 \hat{c}_i \hat{c}_{i+1} + J_2 \hat{c}_{i+1}^+ \hat{c}_i^+ - 2 \delta \hat{c}_i \hat{c}_i^+ \right).$$

This Hamiltonian can be diagonalized using Bogolyubov transformation.
Solution 1: Let us Fourier transform our creation-annihilation operators:

\[ C_m = \int_{-\pi}^{\pi} e^{ikm} \, C_k \, dk \, \Delta \, \text{Then} \]

\[ \sum_m C_m^* C_{m+1} = \sum_k C_k^* C_k e^{ik} \]

\[ \sum_m C_m C_{m+1} = \sum_k C_k C_{-k} e^{-ik} \]

\[ \sum_m (C_m^* C_{m+1} + C_{m+1}^* C_m) = \sum_k 2 C_k^* C_k e^{ik} \]

The \[ \sum_k = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \] is because of anticommutativity of \( C_k \) and \( e^{-ik} \)

\[ \sum_m C_m C_{m+1} = -i \sum_k \sin K C_k C_{-k} \]

After this, our Hamiltonian becomes

\[ \mathcal{H} = \sum_k \left[ z \left( \cos k - \frac{1}{2} \right) C_k^+ C_k - i \tilde{J}_2 \sin K C_k C_{-k} + i \tilde{J}_2 \sin \kappa C_k^+ C_k \right] \]

One can perform a rotation \( C_k = e^{i\frac{\pi}{4} b_k} \)

\[ C_k^+ = e^{-i\frac{\pi}{4} b_k^+} \]

This way so that the Hamiltonian becomes
real: \[ \hat{H} = \sum_k \left( (\hat{S}_1 \cos k - \hat{S}_2 \sin k) b_k^+ b_k + \hat{S}_2 \sin k b_k b_k^+ \text{h.c.} \right) \]

Let us look for fermionic Bogoliubov transformation in the form

\[ b_k = u_k \bar{c}_k + v_k \bar{c}_k^+, \quad b_k^+ = u_k \bar{c}_k^+ + v_k \bar{c}_k \]

\[ b_{-k} = -v_k \bar{c}_k + u_k \bar{c}_k^+, \quad b_{-k}^+ = -v_k \bar{c}_k^+ + u_k \bar{c}_k \]

with real \( u_k \) and \( v_k \), where \( u_k^2 + v_k^2 = 1 \).

This gives us

\[ \hat{H} = \sum_k \left( (\hat{S}_1 \cos k - \hat{S}_2 \sin k) \left( u_k^2 \bar{c}_k^+ \bar{c}_k + u_k v_k (\bar{c}_k^+ \bar{c}_k - \bar{c}_k \bar{c}_k^+) \right) + \right. \]

\[ \left. + u_k^2 \bar{c}_{-k} \bar{c}_{-k}^+ \right) + \]

\[ + \hat{S}_2 \sin k \left( u_k^2 \bar{c}_k \bar{c}_k - u_k v_k (\bar{c}_k^+ \bar{c}_k - \bar{c}_k \bar{c}_k^+) \right) - \]

\[ - u_k^2 \bar{c}_{-k}^+ \bar{c}_{-k}^+ \right) + \text{h.c.} \]

Now let us look for \( u_k, v_k \) such that the coefficient of \( \bar{c}_k \bar{c}_k^+ \) is \( = 0 \). Then
\[ 2 u_k v_k (J_i \cos k - \mu) + (-u_k^2 + v_k^2) J_2 \sin k = 0. \]

This gives us

\[ v_k = \frac{J_i - J_2 \sin k \pm \sqrt{(J_i \cos k - \mu)^2 + J_2^2 \sin^2 k}}{J_2 \sin k}. \]

Solving this with \( u_k^2 + v_k^2 = 1 \) and substituting the result into \( \hat{\mathbf{A}} \) gives.

\[ \hat{\mathbf{A}} = \text{const} + \sum_k \left[ 2 (J_i \cos k - \mu) (u_k^2 - v_k^2) + 4J_2 \sin k u_k v_k \right] c_k^+ c_k = \]

\[ = \text{const} + \sum_k \varepsilon_k c_k^+ c_k, \]

where

\[ \varepsilon_k = 2 \sqrt{(J_i \cos k - \mu)^2 + J_2^2 \sin^2 k}. \]

At absence of dispersion at \( J_i = J_2, \mu = 0 \)

means that excitation are localized within one or a few sides of the chain.
Solution 2: Let us consider the Fourier transformed form of the Hamiltonian:

\[ \hat{H} = \sum_k \left[ 2 (J_1 \cos k - J_2) \hat{c}_k \hat{c}^*_k - iJ_2 \sin k \hat{c}_k \hat{c}^*_k + iJ_2 \sin k \hat{c}^*_k \right] \]

Then, one can compute a commutator \([\hat{H}, \hat{c}_k]\)

using the following identities:

\[ [\hat{c}^*_k, \hat{c}_k] = \hat{c}^*_k \hat{c}_k - \hat{c}_k \hat{c}^*_k = -\hat{c} \]

\[ [\hat{c}^*_k, \hat{c}^*_k] = -\hat{c}^* \]

One will thus obtain

\[ [\hat{H}, \hat{c}_k] = -2 (J_1 \cos k - J_2) \hat{c}_k + 2iJ_2 \sin k \hat{c}^*_k \]

Let us remember that the physical meaning of the commutator of a given operator with \(\hat{H}\) is

\[ it\hat{A} = [\hat{A}, \hat{H}] \rightarrow \text{is the "speed of change" of } \hat{A}. \]

Now let us consider canonical transformation

\[ \hat{c}_k = \hat{u}_k \hat{b}_k + \hat{v}_k \hat{b}^*_k, \quad \hat{c}^*_k = \hat{u}_k^* \hat{b}^*_k - \hat{v}_k^* \hat{b}_k, \]
and let us assume that parameters $u_k, v_k$ are chosen in such a way that the Hamiltonian is diagonal in $b_k, b_k^+$:

$$\hat{H} = E_0 + \sum_k \varepsilon_k b_k^+ b_k.$$

Now let us compute the following commutators:

$$[\hat{H}, b_k] = -\varepsilon_k b_k, \quad [\hat{H}, b_k^+] = \varepsilon_k b_k^+.$$

Using these expressions, we can rewrite the commutator $[\hat{H}, c_k]$ as follows:

$$[\hat{H}, c_k] = \left[ E_0 + \sum_k \varepsilon_k b_k^+ b_k, c_k \right] = -u_k \varepsilon_k b_k + v_k \varepsilon_k b_k^+.$$

On the other hand, the RHS of the expression of $[\hat{H}, c_k]$ on page 15 will acquire the form:

$$[\hat{H}, c_k] = -u_k \varepsilon_k b_k + v_k \varepsilon_k b_k^+ = -2 \left( J_1 \delta_{k,k'} - J \right) (u_k b_k + v_k b_k^+)$$

$$+ 2 J_2 i \sin k (u_k^* b_k^+ - v_k^* b_k).$$
From here one obtains equations for $E_k, U_k, V_k$:

\[ U_k E_k = 2 \left( J_1 \cos K - J_2 \right) U_k + 2J_2 i \sin K V_k^* \quad (1) \]

\[ V_k E_k = -2 \left( J_1 \cos K - J_2 \right) V_k + 2J_2 i \sin K U_k^* \quad (2) \]

Since these equations are homogeneous, one can solve (2) for $V_k$:

\[ V_k = \frac{2J_2 i \sin K V_k^*}{E_k + 2 \left( J_1 \cos K - J_2 \right)} \]

and then substitute the solution into (1):

\[ E_k^2 - 4 \left( J_1 \cos K \right)^2 = 4J_2^2 \sin^2 K \]

From here:

\[ E_k = 2 \sqrt{\left( J_1 \cos K - J_2 \right)^2 + J_2^2 \sin^2 K} \]

Notice that we choose the sign, as the excitation energy above the ground state is always positive.

Parameters $U_k, V_k$ can be found from the above equation noting that $U_k^2 + V_k^2 = 1$ under normalization condition.
Jordan-Wigner transformation

A non-interacting gas of fermions is still highly correlated due to the exclusion principle. This is not exploited in the Jordan-Wigner representation of spins.

A classical spin is a vector - a good representation for quantum spins with large $S$. At small $S$, one:

First (naive) attempt

$|S \rangle = \frac{1}{\sqrt{2}} |1\rangle \pm |0\rangle$ => these states can be thought of as empty and singly occupied fermion states:

$|1\uparrow \rangle \equiv |f^+ 0\rangle$, $|1\downarrow \rangle \equiv |0 \rangle$

An explicit representation of the spin-raising and spin-lowering operators is then:

$s^+ = f^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $s^- = f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

For $z$-component thus one has:

$s_z = \frac{1}{2} \left( |1\uparrow \rangle \langle 1\uparrow | - |1\downarrow \rangle \langle 1\downarrow | \right) = f^+ f - \frac{1}{2}$

Reconstructing transverse spin operators we see:

$s_x = \frac{1}{2i} (s^+ - s^-) = \frac{1}{2i} (f^+ f - f^- f)$

Majorana fermions:

$\tilde{s} = \frac{1}{2} (f^+ + f) \Rightarrow \tilde{s} = \frac{1}{2} (f^+ f - f^- f)$

$\bar{s} = \frac{1}{2} (f^+ - f - f^+ f) \Rightarrow \bar{s} = \frac{1}{2i} (f^+ f - f^- f)$

$\bar{\tilde{s}} = \frac{1}{2i} (f^+ + f)$
The explicit matrix representation of these operators makes it clear that
\[
[\hat{S}_a, \hat{S}_b] = i \epsilon_{abc} \hat{S}_c - \text{ the correct algebra } \mathfrak{su}(2)
\]
however, due to "fermionic" nature of \( S \)
\[
\{ \hat{S}_a, \hat{S}_b \} = \frac{1}{4} \{ \hat{S}_a, \hat{S}_b \} = \frac{1}{2} \delta_{ab}, \quad \hat{S}_a = \frac{1}{2} \hat{g}_a, \quad a = 1, 2, 3
\]
where \( \hat{g}_1 = \hat{g}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{g}_2 = \hat{g}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{g}_3 = \hat{g}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\)
are Pauli matrices with properties \( \text{Tr} \hat{g}_i = 0 \) and \( \det \hat{g}_i = 1. \)

Full fermionization in 1D:

If there is more than one spin in the system, we know that:
* independent spin operators commute.
* independent fermions anticommute.

For Jordan-Wigner transform: to fix this in 1D,
let us attach a string to the fermion at site \( j \)
\[
\hat{S}_j^+ = \hat{f}_j^+ e^{i \phi_j}, \quad \text{where}
\]
the phase operator \( \phi_j \) contains the sum over all fermion occupancies of sites to the left of \( j \):
\[
\phi_j = \bar{u} \sum_{k < j} \hat{n}_k, \quad e^{i \phi_j} - \text{is called a string operator.}
\]
The complete transform is then
\[ \hat{S}_j^2 = \hat{f}_j^+ \hat{f}_j - \frac{1}{i^2} \]
\[ \hat{S}_j^+ = \hat{f}_j^+ e^{i\sum \frac{\varepsilon_i}{e^i}} \hat{h}_j \]
\[ \hat{S}_j^- = \hat{f}_j e^{-i\sum \frac{\varepsilon_i}{e^i}} \hat{h}_j \]
where \( e^{i\eta} = e^{-i\eta} \), is a Hermitian operator.

Important property: the string operator \( e^{i\phi} \) anticommutes with any fermion operator to the left of its free end.

\[ \begin{array}{cccc}
1 & 2 & 3 & \hat{S}_4^{(+)} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow & \uparrow \\
N & N & N & N \\
\end{array} \]

proof: a) let us see that \( e^{i\eta} \) anticommutes with \( \hat{f}_j \):
\[ \begin{align*}
\{ e^{i\eta}, f_j \} &= e^{i\eta} f_j + f_j e^{i\eta} \\
e^{i\eta} &= 1 + i\eta + \frac{1}{2} (i\eta)^2 + \cdots = 1 + i\eta + \frac{n}{2} (i\eta)^2 + \cdots + \frac{n}{e!} (i\eta)^n \\
&= 1 + n \left( \sum_{\ell=0}^{\infty} \frac{(i\eta)^\ell}{e!} - 1 \right) = 1 + n \left( e^{i\eta} - 1 \right) = 1 - 2n \\
\Rightarrow \{ e^{i\eta}, f_j \} &= (1 - 2n) f_j + f_j (1 - 2n) = -2 f_j f_j^+ f_j + 2 f_j f_j^+ f_j + 2 f_j = 0.
\end{align*} \]
and similarly, \( e^{i \phi_l} f_{i}^+ f_{j}^+ f_{k}^+ f_{l}^+ = 0 \).

b) Now, the phase \( e^{i \phi_i} \) at any \( l \neq i \) commutes with \( f_i \) and \( f_i^+ \), the string operator \( e^{i \phi_i} \) anticommutes with all fermions at all sites \( l \) to the left of \( i \):
\[
\left[ e^{i \phi_i}, f_{e}^{(+)} \right] = 0, \quad l < i
\]
while it commutes with fermions at all other sites:
\[
\left[ e^{i \phi_i}, f_{e}^{(+)} \right] = 0, \quad e \geq i.
\]

Let us now verify that the transverse spin operators satisfy the correct commutation algebra.

Suppose \( j < k \) \( \Rightarrow \) \( e^{i \phi_j} \) commutes with fermions at sites \( j \) and \( k \) \( \Rightarrow \)

\[
\left[ S_{j}^{+}, S_{k}^{+} \right] = \left[ f_{j}^{(+)} e^{i \phi_j}, f_{k}^{(+)} e^{i \phi_k} \right] =
\]
\[
e^{i \phi_j} \left[ f_{j}^{(+)} e^{i \phi_j}, f_{k}^{(+)} e^{i \phi_k} \right] =
\]
Here, \( f_{j}^{(+)} \) anticommutes with both, \( f_{k}^{(+)} \) and \( e^{i \phi_k} \) \( \Rightarrow \) it commutes with their product \( f_{k}^{(+)} e^{i \phi_k} \)
\[
\Rightarrow \left[ S_{j}^{+}, S_{k}^{+} \right] \equiv e^{i \phi_j} \left[ f_{j}^{(+)} f_{k}^{(+)} e^{i \phi_k} \right] = 0
\]
\[\text{--}\]
Example: 1D Heisenberg chain \((XXZ)\)

\[
H = -J \sum \left[ S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right] - J_z \sum_j S_j^z S_{j+1}^z.
\]

Can be rewritten as:

\[
H = -\frac{J}{2} \sum \left[ S_j^{+} S_{j+1}^{-} + h.c. \right] - J_z \sum_j S_j^z S_{j+1}^z.
\]

Fermionization yields:

\[
\frac{J}{2} \sum_j S_j^{+} S_j^{-} = \frac{J}{2} \sum_j f_j^{+} f_j^{-} \quad f_j^{+} = \frac{J}{2} \sum f_{j+1}^{+} f_j^{-}
\]

For the \(z\)-component, one obtains:

\[
-J_z \sum_j S_j^z S_{j+1}^z = -\frac{J_z}{2} \sum_i \left( n_{j+1} - \frac{1}{2} \right) \left( n_j - \frac{1}{2} \right),
\]

and thus ferromagnetic spin-interaction means that spin fermions attract each other.

Fermionized Hamiltonian thus reads:

\[
H = -\frac{J}{2} \sum_j \left( f_j^{+} f_j^{-} + f_j^{+} f_{j+1}^{-} \right) + \sum_j n_j - J_z \sum_j n_j n_{j+1}
\]

While fermionization of \(XYZ\) model:

\[
\hat{H}_{XYZ} = -\sum_{i=0}^{\infty} \left( \sum_{j=1}^{\infty} \left( J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z \right) \right) - J S_i^z
\]

gives:

\[
\sum \hat{H}_{XYZ} = \sum_{i=0}^{\infty} \left( J_1 C_i^{+} C_{i+1}^{+} + J_2 C_i C_{i+1} + h.c. \right) - J (n_i - \frac{1}{2}) + J_2 (n_i - \frac{1}{2})(n_{i+1} - \frac{1}{2})
\]

\[
J_2 = \frac{\sqrt{J_x^2 + J_y^2}}{4}
\]