Here we need to find function \( f(x) \), that minimizes the action:

\[
S[f] = \int_{x_i}^{x_f} g(f, f', x) \, dx , \quad \text{where} \quad g = \sqrt{1 + [f'(x)]^2}.
\]

Step 1:

\[
\frac{\partial g}{\partial f} = 0.
\]

Step 2:

\[
\frac{\partial g}{\partial f'} = \frac{1}{2\sqrt{1 + (f')^2}}, \quad \text{so} \quad f' = \frac{f'}{\sqrt{1 + (f')^2}}.
\]

Step 3:

\[
\frac{d}{dt} \left( \frac{2g}{2f'} \right) = \frac{d}{dt} \left( \frac{f'}{\sqrt{1 + (f')^2}} \right) = \frac{f''}{\sqrt{1 + (f')^2}} - \frac{2(f')^2 f''}{2(1 + (f')^2)^{3/2}} = \frac{f'' (1 + (f')^2) - (f')^2 f''}{(1 + (f')^2)^{3/2}} = \frac{f''}{(1 + (f')^2)^{3/2}} - \frac{f''}{[1 + (f'(x))^2]^{3/2}}.
\]

From Euler-Lagrange equations we obtain

\[
\frac{d}{dt} \left( \frac{2g}{2f'} \right) = 0 \implies \frac{f''}{[1 + (f'(x))^2]^{3/2}} \Rightarrow f''(x) = 0.
\]

Therefore \( f(x) = ax + b \) is a linear function connecting \((x_i, y_i)\) and \((x_f, y_f)\) points.
Problem 2:
Choose \((r, \theta)\) as generalized coordinates.

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  \dot{x} &= \dot{r} \cos \theta - r \dot{\theta} \sin \theta \\
  \dot{y} &= \dot{r} \sin \theta + r \dot{\theta} \cos \theta
\end{align*}
\]

Lagrangian is \(L = T - U\),

where kinetic energy \(T\) is given by

\[
T = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right) = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right).
\]

Potential energy \(U\) can be found from

\[
F = -\frac{\partial U}{\partial r}, \quad \text{where} \quad F = -A \cdot r^{d-1}
\]

From here \(U = -\int F \, dr = A \int r^{d-1} \, dr = A \frac{r^d}{d}\).

Therefore

\[
L = T - U = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - A \frac{r^d}{d}
\]

Euler Lagrange equations:

For \(r\):
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \implies m \ddot{r} - mr \dot{\theta}^2 + Ar^{d-1} = 0
\]

For \(\theta\):
\[
\frac{d}{d\theta} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \implies \frac{d}{dt} \left( mr^2 \dot{\theta} \right) = 0.
\]
b) Since \( l_0 = m r^2 \dot{\theta} \), we see that the Euler-Lagrange equation for \( \theta \) gives \( \frac{d}{dt} (l_0) = 0 \), which implies that the angular momentum \( l_0 \) is conserved.

c) Since \( l_0 = m r^2 \dot{\theta} = \text{const. (time independent)} \), we can solve this condition for \( \dot{\theta} \):

\[
\dot{\theta} = \frac{l_0}{m r^2}
\]

and substitute this expression into Euler-Lagrange eq. for \( r \):

\[
m \ddot{r} - m r \dot{\theta}^2 + A r^{d-1} = 0
\]

From here we get:

\[
m \ddot{r} - \frac{l_0^2}{m r^3} + A r^{d-1} = 0
\]

*Note: This equation is equivalent to the condition of energy conservation: \( \frac{d}{dt} (T + U) = 0 \).
Problem 3: Solution

\[ U(r) = -Ve^{-kr}, \quad V > 0, \quad k > 0. \]

The effective potential is given by

\[ V_{\text{eff}}(r) = U(r) + \frac{\ell_0^2}{2mr^2} = -Ve^{-kr} + \frac{\ell_0^2}{2mr^2}. \]

A bounded motion is possible if \( V_{\text{eff}}(r) \) has a minimum: \( \frac{dV_{\text{eff}}}{dr} = 0 \) at \( r = r_0 \).

\[ V_{\text{eff}}' = Vke^{-kr} - \frac{\ell_0^2}{mr_0^2} = 0 \quad \Rightarrow \quad Vkr_0^3 e^{-kr_0} = \frac{\ell_0^2}{m} \]

\[ \Rightarrow (kr)^3 e^{-kr} = \frac{\ell_0^2 k^2}{mV}, \quad \text{so} \quad f(x) = \frac{\ell_0^2 k^2}{mV} \quad \text{where} \]

we choose introduce a new dimensionless variable \( X = kr \). Examine \( f(x) = x^3 e^{-x} \):

\[ f(0) = 0, \quad f'(x_0) = 0 \Rightarrow x_0 \text{ is maximum} \]

\[ f'(x_0) = 3x_0^2 e^{-x_0} - x_0^3 e^{-x_0} = 0 \]

\[ x_0^2 e^{-x_0} (3 - x_0) = 0 \]

So \( x_0 = 3 \) corresponds to maximum of \( f(x) \).
bounded motion takes place here.

we have a condition: \( f(x) = \frac{lo^2 k^2}{mV} = \text{const} \)

which has a solution if

\[
\frac{lo^2 k^2}{mV} \leq f(x_0)
\]

\[
f(x_0) = f(3) = 3^3 \cdot e^{-3} = \frac{27}{e^3}
\]

so the condition is

\[
\frac{lo^2 k^2}{mV} \leq \frac{27}{e^3} \quad \Rightarrow \quad lo^2 \leq \frac{27mV}{e^3 k^2}
\]