Homework 8

HW problem 1: Principle of the least action. A point particle moves in one dimension in the potential field, $U(x) = -Fx$. It starts its motion from the origin of the coordinate system $x(t = 0) = 0$ and arrives at $x(t = \tau) = a$ in time $\tau$. Find $x(t)$, assuming that it has the following form: $x(t) = At^2 + Bt + C$.

Hints:
1. Lagrangian describing the system is given by $\mathcal{L}(x, \dot{x}) = T - U$, where $T$ is the kinetic energy of the particle, while $U$ is the potential.
2. Compute the action $S = \int_0^\tau \mathcal{L}(x, \dot{x}, t) dt$. Determine parameters $A$, $B$, and $C$ from the principle of the least action (find those values of $A$, $B$, $C$, for which $S$ is minimal). After finding the solution for $x(t)$, check that the solution satisfies Euler-Lagrange equations.

HW problem 2: Find the equation of motion of a one-dimensional system described by the following Lagrangian: $\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}e^{\alpha t} (\dot{x}^2 - \omega^2 x^2)$. After deriving the equation comment on what system is this?

Hint: The equation of motion is found upon substitution of $\mathcal{L}(x, \dot{x}, t)$ into Euler-Lagrange equation:
\[
\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0.
\] (1)

HW problem 3: We showed in class that if a smooth curve $y = f(x)$ in the $xy$-plane passes through the two given points $(x_{in}, y_{in})$ and $(x_f, y_f)$, then the length of the line between these two points is
\[
S[f] = \int_{x_{in}}^{x_f} g(f, f', x) dx \quad \text{where} \quad g = \sqrt{1 + (f'(x))^2}.
\] (2)

Show that minimization of the functional Eq. (2) yields a linear function $f(x) = ax + b$. Discuss the implication of this property. Why does this mean that the shortest path between initial and final points is the straight line that connects them with each other?

Hint: Solve the following Euler-Lagrange equation for $f(x)$:
\[
\frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'} = 0.
\] (3)