Outline:

* Small (coupled) oscillations
  - general approach

* Example: application to the system of two coupled harmonic oscillators.
Lagrangian approach to mechanics

is also useful to study the small oscillations of systems with many degrees of freedom.

From definition we have that the potential is given by:

\[ V(q_i) = -\mathbf{L}(\dot{q}_i = 0, q_i). \]

\( V \) is obtained by setting to zero the generalized velocities in the Lagrangian.

The positions of equilibrium are found from the potential imposing the condition that the derivatives of the potential w.r.t. the generalized coordinates vanish, namely:

\[ \frac{\partial V}{\partial q_i} |_{\dot{q}_i = \bar{q}_i} = 0, \]

where \( \bar{q}_i, i = 1, \ldots, DGF \) are the values corresponding to the minimum. (Note that the second derivative test is required \( \frac{\partial^2 V}{\partial q_i \partial \dot{q}_i} > 0 \) to insure that found points are stable equilibrium.)
To study small oscillations about the equilibrium point, we define new coordinate 

\( \delta q_i(t) = q_i(t) - \bar{q}_i \) - coordinate relative to the equilibrium that we treat as "small": \( \delta q_i(t) \ll q_i(0) \).

Then we have 

\[ L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - V(q_i) \]

with 

\[ V(q_i) = V(\bar{q}_i + \delta q_i) = \]

\[ V(\bar{q}_i) + \sum_{j} \frac{\partial V}{\partial \delta q_j} \delta q_j + \frac{1}{2} \sum_{jk} \frac{\partial^2 V}{\partial \delta q_j \partial \delta q_k} \delta q_j \delta q_k + O(\delta q)^3 = \]

\[ V(\bar{q}_i) + \frac{1}{2} \sum_{jk} \frac{\partial^2 V}{\partial \delta q_j \partial \delta q_k} \delta q_j \delta q_k, \]

since \( \frac{\partial V}{\partial q_i} \mid_{q_i = \bar{q}_i} = 0 \) by definition of \( \bar{q}_i \)-equilibrium point.

So we have an expression for \( V(q_i) \). As for kinetic energy \( T \), we use the form that is quadratic in \( (\delta \ddot{q}_i) \) to arrive at 

\[ T = \frac{1}{2} \sum_{jk} \frac{\partial^2 T}{\partial \delta q_j \partial \delta q_k} \mid_{\dot{q}_i = \bar{q}_i} \delta \ddot{q}_j \delta \ddot{q}_k. \]
Therefore, the Lagrangian acquires the following, "quadratic" in $\delta q_i$'s and $\delta \dot{q}_i$'s form:

$$\mathcal{L} = \frac{1}{2} \sum_{ij} T_{ij} \delta q_i \delta \dot{q}_j - \frac{1}{2} \sum_{ii} V_{ii} \delta q_i \delta q_i + O(\delta q_i)^3 - V(q_i),$$

where $T_{ij} = \frac{\partial^2 T}{\partial q_i \partial q_j}$ evaluated at $q_i = \bar{q}_i$, $\dot{q}_i = \ddot{q}_i$,

and $V_{ii} = \frac{\partial^2 V}{\partial q_i \partial q_i}$ evaluated at $q_i = \bar{q}_i$.

Sets of $T_{ij}$, $V_{ii}$ are constant numbers, with obviously $T_{ij} = T_{ji}$, $V_{ii} = V_{ii}$.

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Note: In order to make sure that $q_i = \bar{q}_i$ is actually a minimum rather than a maximum or a saddle point, we must impose that the matrix whose elements are $V_{ii}$ is positive definite, i.e. that the eigenvalues of $V_{ii}$ are all positive.
That in a given situation and for given $T$ and $V$, one can compute both sets: $T_{ij}$ and $V_{ij}$ of mass constants. This odd $f$ yields a quadratic in $\delta q^i$ and $\delta \dot{q}^i$'s form of the Lagrangian, from where one can find the equations of motion for $\delta q^i, \quad i = 1, \ldots, \# DOF.$

So we write the equations of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad \text{which gives}$$

$$\frac{d}{dt} \left( \sum_{j} T_{ij} \delta \dot{q}^j \right) = - \sum_{j} V_{ij} \delta q^i, \quad \text{or}$$

$$\sum_{i} T_{ij} \delta \ddot{q}^i = - \sum_{j} V_{ij} \delta q^i, \quad i = 1, \ldots, \# DOF.$$

Notice that these equations look like the equation for a harmonic oscillator.

So we look for a solution in the form

$$\delta q^i(t) = s_i e^{i\omega t}, \quad \text{where } s_i \text{ and } \omega$$

are (yet unknown) constants.
Substituting this ansatz for \( s \) into the equation of motion (4) on page 5, we arrive at:

\[- \sum_i T_{ij} w^2 S_j = - \sum_i V_{ij} S_i.\]

As the next step we introduce a pair of symmetric matrices \( \Omega \) and \( \Omega^\dagger \), where

\[ (\Omega)_{ij} = T_{ij} \quad \text{and} \quad (\Omega^\dagger)_{ij} = V_{ij} \]

are entries of matrices \( \Omega \) and \( \Omega^\dagger \).

Also, let us denote by \( \Omega^\dagger \) the vector

\[ \Omega^\dagger = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix}, \]

where \( n \) is the \# of degrees of freedom in our system.

Then our equation of motion acquires the following matrix form:

\[ (\Omega - w^2 \Omega), \Omega^\dagger S = 0. \]

This reminds us of the equation for the eigenvalues of a matrix.
In order for the equation
\[(\overline{\mathbf{v}} - \omega^2 \overline{\mathbf{I}}) \mathbf{s} = \mathbf{0}\]
to have a nonzero solution, we need
\[\det[\overline{\mathbf{v}} - \omega^2 \overline{\mathbf{I}}] = 0.\]

This gives an equation for \(\omega^2\); generally, since \(\overline{\mathbf{v}}\) and \(\overline{\mathbf{I}}\) are \(n \times n\) matrices, we find as many values of \(\omega^2\) as the number of degrees of freedom, \(n: \omega_1^2, \omega_2^2, \ldots, \omega_n^2.\)

Note: it is possible to prove that if \(\mathbf{v}_{ij}\) is positive definite, then all \(\omega_i^2\) are real and positive.

Then for every \(\omega_k^2\), we find a corresponding vector \(\mathbf{s}_k.\)

The general solution will thus be
\[
\begin{pmatrix}
\mathbf{s}_{p1} \\
\vdots \\
\mathbf{s}_{pn}
\end{pmatrix} = \sum_{k=1}^{n} \mathbf{c}_k \mathbf{s}_k \mathbf{e}^{i\omega_k t} = \\
\begin{pmatrix}
\sum_k \mathbf{c}_k (\mathbf{s}_k)_{1} e^{i\omega_k t} \\
\vdots \\
\sum_k \mathbf{c}_k (\mathbf{s}_k)_{n} e^{i\omega_k t}
\end{pmatrix}
\]
Here, \((c, k, S)\) are arbitrary constants, which can be fixed e.g. from initial conditions.

**Application:** let us apply this theory to the following example: two coupled harmonic oscillators.

\[
\begin{align*}
\text{The system consists of 3 springs with constants } k_1, k_2, k_3 \text{ attached to masses } m_1 \text{ and } m_2. \\
\text{The total distance } l \text{ is fixed.}
\end{align*}
\]

Then the potential is:

\[
V(x_1, x_2) = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (l - x_2)^2
\]

and kinetic energy is:

\[
T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2}, \text{ so that the Lagrangian acquires the form:}
\]

\[
L = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2 - \frac{1}{2} k_3 (l - x_2)^2.
\]
Equilibrium positions:

\[ 0 = \frac{\partial V}{\partial x_1} = x_1 \frac{k_1}{k_3} + k_2 (x_1 - x_2) \]

\[ 0 = \frac{\partial V}{\partial x_2} = k_2 (x_2 - x_1) + k_3 (x_2 - l) \]

Solving for \( x_1 \) and \( x_2 \) we find:

\[ \bar{x}_1 = \frac{\frac{l}{k_3}}{\left(\frac{k_1}{k_3}\right) + 1 + \left(\frac{k_2}{k_3}\right)} \quad \text{and} \quad \bar{x}_2 = \bar{x}_1 \cdot \left(1 + \frac{k_1}{k_2}\right) \]

For simplicity, let us take \( k_1 = k_2 = k_3 \) and \( m_1 = m_2 = m \). Then \( \bar{x}_1 = \frac{l}{3} \), \( \bar{x}_2 = \frac{2}{3} l \).

Defining \( \delta x_1 = x_1 - \frac{l}{3} \) and \( \delta x_2 = x_2 - \frac{2}{3} l \) displacement w.r.t. equilibrium

\[ \Rightarrow \quad x_1 = \frac{l}{3} + \delta x_1 \quad \text{and we obtain} \]

\[ x_2 = \frac{2}{3} l + \delta x_2 \]

\[ L = \frac{m \dot{x}_1^2}{2} + \frac{m \dot{x}_2^2}{2} - \frac{k}{2} \left( x_1^2 + x_2^2 - 2x_1x_2 + x_1^2 + l^2 - 2lx_2 + x_2^2 \right) = \]

\[ = \frac{m \delta \dot{x}_1^2}{2} + \frac{m \delta \dot{x}_2^2}{2} - \frac{k}{2} \left[ \frac{l^2}{3} + 2 \delta x_1^2 + 2 \delta x_2^2 - 2 \delta x_1 \delta x_2 \right] \]
Therefore matrices $\bar{T}$ and $\bar{V}$ have the following form:

$$\bar{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$  

Finally, the angular frequencies are found by solving the characteristic equation:

$$\det[\bar{V} - \omega^2 \bar{T}] = 0 \Rightarrow$$

$$\Rightarrow \det \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} = 0$$

Therefore:

$$(2k - \omega^2 m)^2 - k^2 = 0$$

$$2k - \omega^2 m = \pm k$$

$$\omega^2 = \frac{2k \pm k}{m} \Rightarrow \omega^2 = \frac{3k}{m}$$

$$\omega_+ = \sqrt{\frac{3k}{m}}, \quad \omega_- = \sqrt{\frac{k}{m}}.$$  

Therefore the frequencies are

$$\omega_+ = \sqrt{\frac{3k}{m}}, \quad \omega_- = \sqrt{\frac{k}{m}}.$$
What motion do these angular frequencies correspond to? Our ansatz for the solution $\delta x_1(t), \delta x_2(t)$ was:

$$
\begin{pmatrix}
\delta x_1(t) \\
\delta x_2(t)
\end{pmatrix} = \mathcal{S} e^{i\omega t}.
$$

Since our system is linear, the linear combination of solutions is a solution.

In general:

$$
\begin{pmatrix}
\delta x_1(t) \\
\delta x_2(t)
\end{pmatrix} = \text{Re} \left[ e^{i\omega t} + e^{-i\omega t} \mathcal{S}_+ e^{i\omega t} + e^{-i\omega t} \mathcal{S}_- e^{i\omega t} \right],
$$

where \(\mathcal{S}_\pm\) are arbitrary complex numbers, and \(\mathcal{S}_\pm\) are solutions of \((\mathcal{V} - \omega^2 \mathcal{T}) \mathcal{S}_\pm = 0\).

In our case, \(\mathcal{S}_+\) satisfies \((\mathcal{V} - \omega^2 \mathcal{T}) \mathcal{S}_+ = 0\).

This means:

$$
\begin{pmatrix}
2k - \frac{3k}{m} & -k \\
-k & 2k - \frac{3k}{m}
\end{pmatrix}
\begin{pmatrix}
(S_+)_1 \\
(S_+)_2
\end{pmatrix} = 0
$$

$$
\Rightarrow -k \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} (S_+)_1 \\ (S_+)_2 \end{pmatrix} = 0 \text{ gives } (S_+)_1 + (S_+)_2 = 0
$$

$$
\Rightarrow \mathcal{S}_+ = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$
Now, since \( (\delta x_1) = e^{i\omega t} (1) \) and \( (\delta x_2) = e^{i\omega t} (-1) \), the motion with frequency \( \omega \) corresponds to

\[ \delta x_1(t) = -\delta x_2(t) \]

oscillations in opposite directions.

For motion with \( \omega \) we look for a solution for \( \tilde{s}_- \):

\[
\begin{pmatrix}
2k - \frac{k}{M} & -k \\
-k & 2k - \frac{k}{M}
\end{pmatrix}
\begin{pmatrix}
(\tilde{s}_-)_1 \\
(\tilde{s}_-)_2
\end{pmatrix} = 0
\]

or

\[
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
(\tilde{s}_-)_1 \\
(\tilde{s}_-)_2
\end{pmatrix} = 0
\]

giving \( (\tilde{s}_-) = (\tilde{s}_-)_2 \) = \( \beta (1) \).

Therefore the motion with frequency \( \omega \) will correspond to \( \delta x_1 = \delta x_2 \).