Lecture 8

Outline:

- Principles of superposition: Fourier series
- Conservative quantities and conservative systems
Principles of superposition - Fourier series

The forced oscillations we have been discussing obey the differential equation

\[ \left( \frac{d^2}{dt^2} + \alpha \frac{d}{dt} + \omega^2 \right) x(t) = A \cos \omega t \]

Linear operator = \( L \).

Define a linear operator, \( \hat{L} = \frac{d^2}{dt^2} + \alpha \frac{d}{dt} + \omega^2 \).

If satisfies

\[ \begin{align*}
\hat{L} [f(t)+g(t)] &= \hat{L} f(t) + \hat{L} g(t) \quad \text{principle of superposition} \\
\hat{L} (c \cdot f(t)) &= c \cdot \hat{L} f(t) \quad , \quad c = \text{const.}
\end{align*} \]

Therefore, if we have two solutions \( x, x_2 \) such that \( L x_1 = F_1(t) \), \( L x_2 = F_2(t) \), then

\[ L (\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F_1(t) + \alpha_2 F_2(t). \]

We can extend this to a set of solutions \( x_n(t) \):

\[ L \left( \sum_{n=1}^{N} a_n x_n(t) \right) = \sum_{n=1}^{N} a_n F_n(t) \]
If we identify our equation as

\[ x(t) = \sum_{n=1}^{\infty} a_n x_n(t) \]

\[ F(t) = \sum_{n=1}^{\infty} a_n F_n(t), \]

then \[ L x(t) = F(t). \]

If each \( F_n(t) \) has a simple harmonic dependence on \( t \): \( \omega_n t + \phi_n \), then

\[ F(t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t - \phi_n). \]

The steady state solution is

\[ x(t) = \frac{1}{\text{Im}} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\left(\omega_n - \omega_0\right)^2 + 4\omega_n^2 \beta^2}} \cos(\omega_n t - \phi_n - \delta_n), \]

where \( \delta_n = \tan^{-1}\left(\frac{2\omega_n \beta}{\omega_0^2 - \omega_n^2}\right) \).

If \( F \) is periodic \( F(t+\delta) = F(t), \quad \delta = \frac{2\pi}{\omega} \), then

\[ F(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(\omega_n t + b_n \sin(\omega_n t) \right). \]

\[ \text{Fourier Theorem.} \]
where
\[ a_n = \frac{2}{\pi} \int_{\omega t}^{\omega t + \pi} F(t') C_n \cos \omega t' \, dt' \]
\[ b_n = \frac{2}{\pi} \int_{\omega t}^{\omega t + \pi} F(t') \sin \omega t' \, dt' \]

**Part 2:**

Reminder: Consider two isolated bodies f and g.

According to Newton's \( \text{III} \) Law:
\[ \vec{F}_1 = -\vec{F}_2 \implies m_1 a_1 = -m_2 a_2 \]
\[ m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt} \]
\[ m_1 (\vec{v}_1 + m_1 \vec{v}_2) \frac{d\vec{v}_2}{dt} = 0 \]
\[ \Rightarrow \vec{p}_1 + \vec{p}_2 = \text{constant vector} \]
Conservation of momentum
Conservative systems: 1-Dimensions (1D)

In general (arbitrary dimension) a force \( \vec{F}(\vec{r}) \) is called conservative if there is a function \( V(\vec{r}) \) such that

\[
(\star) \quad \vec{F}(\vec{r}) = -\nabla V(\vec{r}) = - \text{grad} V(\vec{r}).
\]

Here \( V(\vec{r}) \) is called potential. In components one has

\[
F_{x_1} = -\frac{\partial V}{\partial x_1}, \quad F_{x_2} = -\frac{\partial V}{\partial x_2}, \quad F_{x_3} = -\frac{\partial V}{\partial x_3}, \quad \ldots
\]

In 1D, all forces that depend on the coordinate \( x \) only, are conservative. This can be seen by solving Eq (\star) for the potential \( V(x) \) as

\[
V(x) = -\int_{x_0}^{x} F(x') \, dx'.
\]

which is a definite integral. Note that by changing the lower integration limit we add a constant to \( V(x) \). Such a constant is physically irrelevant. The potential is defined up to an additive constant.

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Suppose we have a 1D system with force \( F(x) \) and potential \( V(x) \).
The equation of motion reads:

$$m \frac{d\mathbf{v}}{dt} = F(x).$$

We can make use of the trick we applied in solving the problem of the simple harmonic oscillator:

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dx} \cdot \frac{dx}{dt} = \frac{d\mathbf{v}}{dx} \cdot \mathbf{v} = \frac{1}{2} \frac{d}{dx} (\mathbf{v}^2)$$

using $F(x) = -\frac{dV(x)}{dx}$ (V is the velocity, V is the potential).

We have that our equation of motion is equivalent to

$$\frac{d}{dx} \left( \frac{1}{2} m \mathbf{v}^2 \right) = -\frac{dV(x)}{dx} \Rightarrow \frac{d}{dx} \left( \frac{1}{2} m \mathbf{v}^2 + V(x) \right) = 0.$$

We call $\frac{1}{2} m \mathbf{v}^2 \rightarrow \text{Kinetic energy}.$

Therefore we obtain that

$$\frac{1}{2} m \mathbf{v}^2 + V(x) = E = \text{constant}.$$
E is an integration constant, so it depends on the initial conditions.

One can proceed and solve the equation of motion for \( x(t) \) by integrating:

\[
\frac{1}{2} m \left( \frac{dx}{dt} \right)^2 = E - V(x) \quad \text{and we separate}
\]

\[
\int_{x(t_0)}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m} [E - V(x')]}} = \int_{t_0}^{t} dt' = t - t_0.
\]

**Example:** The Harmonic oscillator

Let us use the method described above (energy conservation!) to find \( x(t) \) for an harmonic oscillator.

For simplicity assume that at 
\( t=0, \ x(t=0) = 0, \) and \( x(t=0) = 0. \)

As the first step, we find the potential. We have 
\( F(x) = -kx \) and \( V(x) = -\int F(x') dx'. \)

Integration yields 
\( V(x) = \frac{1}{2} kx^2 \) (a constant that we set to zero).
Therefore $x(t)$ is found from

$$\sqrt{m} \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{\sqrt{E} - \frac{k}{E} (x')^2}} = t - t_0.$$  

Next we factor $\frac{1}{\sqrt{E}}$ out of the integrand on the left-hand side and define a new variable ($y$) such that

$$y^2 = \frac{k}{2E} (x')^2. \Rightarrow y = \sqrt{\frac{k}{2E}} x', \quad dx' = \sqrt{\frac{2E}{k}} dy$$

The equation giving $x(t)$ becomes

$$\sqrt{\frac{m'}{2}} \cdot \frac{1}{\sqrt{E}} \sqrt{\frac{2E}{k}} \int_{0}^{\sqrt{\frac{E}{2k} \cdot x(t)}} \frac{dy}{\sqrt{1-y^2}} = t$$

One more change of variables: $y = \sin \theta, \quad dy = \cos \theta d\theta$

gives the final result

$$t = \sqrt{\frac{m}{k}} \int_{0}^{\arcsin \left( \sqrt{\frac{k}{2E} x(t)} \right)} \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} = \sqrt{\frac{m}{k}} \arcsin \left( \sqrt{\frac{k}{2E} x(t)} \right)$$

So that we find the solution

$$x(t) = \sqrt{\frac{2E}{k}} \sin \left( \sqrt{\frac{k}{m}} t \right)$$
Finally we remember that

\[ \sqrt{\frac{k}{m}} = w \quad \text{and that} \quad \frac{mv^2}{2} + V(x) = E \]

and that at \( x = 0, \ t = 0 \), \( v = v_0 \).

Since \( V(x=0) = 0 \Rightarrow E = \frac{mv_0^2}{2} \)

We thus obtain

\[ x(t) = \sqrt{\frac{2}{k}} \cdot \frac{mv_0^2}{2} \sin(\omega t) = \frac{v_0}{\omega} \sin(\omega t) \]

That is the solution we were looking for.
So the solution of the equations of motion is determined if we are able to compute the primitive of

\[
\frac{1}{\sqrt{\frac{1}{2}m(E-V(x))}}; \quad \text{i.e.} \quad \int_{x(t)}^{x(t') \frac{dx'}{\sqrt{\frac{1}{2}m(E-V(x))}}} = +\infty.
\]

But even if we cannot compute the primitive, we can study qualitatively the motion.

- Let us graph \( V(x) \) vs \( x \) and \( E \):

Since \( \frac{1}{2}mv^2 > 0 \), we know that motion can happen only for \( E > V(x) \). In our example, this is for \( x < x_0, x_1 < x < x_2, \) and \( x > x_3 \).

Points where \( V(x) = E \) are called turning points, because \( V = 0 \) there, i.e. \( V \) changes the sign!
Note in particular that motion between two turning points is periodic.

The period is computed as

\[ T = 2 \int_{x_1}^{x_2} \frac{dx'}{\sqrt{\frac{2}{m} (E-V(x'))}} \]

Factor 2 comes from the fact that one period corresponds to two trips between the turning points \( x_1 \) and \( x_2 \).

Note then in general \( T \) depends on \( E \).

Example of calculation of a period:

The harmonic oscillator \( \Rightarrow \) \( V(x) = \frac{kx^2}{2} \) - the potential energy.

\[ T = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m} (E - \frac{k}{2} x^2)}} = \]
\[
T = 2 \sqrt{\frac{m}{2E}} \cdot \frac{1}{\sqrt{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \frac{kx^2}{2E}}},
\]

where \( x_1 \) and \( x_2 \) are determined by

\[
E = \frac{kx^2}{2} \Rightarrow x_{1,2} = \pm \sqrt{\frac{2E}{k}} - \text{turning points.}
\]

We define a new integration variable \( y = \frac{kx^2}{2E} \)

\[
\Rightarrow x = y \sqrt{\frac{2E}{k}} \Rightarrow y_{1,2} = \pm 1
\]

\[
dx = \sqrt{\frac{2E}{k}} \cdot dy
\]

and

\[
T = 2 \sqrt{\frac{m}{2E}} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}} = 2 \sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}}
\]

Finally, the integral is computed by setting \( y = \cos \theta \)

and we obtain

\[
T = 2 \sqrt{\frac{m}{k}} \cdot \frac{\pi}{2} = \frac{2\pi}{\omega}
\]