Outline of Lecture 4

- Taylor series
- Differential equations: evolution
- Solution of the equation of motion in some specific cases.
Taylor series.

One application of Taylor series is to compute the value of some function in the close vicinity of a point, where the value of this function is known.

Suppose we want to compute \((2.01)^2\). I know that \(2^2 = 4\). So \((2.01)^2\) will be close to

**How close?** In the first approximation one can use the definition of the derivative.

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.
\]

For \(\Delta x\) small but finite I can write

\[
f(x+\Delta x) \approx f(x) + \Delta x \cdot f'(x).
\]

In my example: \(f(x) = x^2\), with \(x = 2\), \(\Delta x = 0.01\).

\[
\Rightarrow f(2.01) \approx f(2) + 0.01 \times f'(2)
\]

with \(f(2) = 2^2 = 4\), \(f'(x) = 2x \Rightarrow f'(2) = 2 \times 2 = 4\)

So \((2.01)^2 \approx 4 + 0.01 \times 4 = 4.04\).

It is actually possible to do better than this.
In general:

\[ f(x+\Delta x) = f(x) + \Delta x \cdot f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(x) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \Delta x^n \]

where \( f^{(n)}(x) \) denotes the \( n \)th derivative of \( f(x) \) and \( f^{(0)}(x) = f(x) \).

In our example: \( f^{(4)}(2) = 2x \), \( f^{(3)}(2) = 2 \), \( f^{(2)}(2) = 0 \)

\[ f(2.01) = 2^2 + 0.01 \cdot (2 \cdot 2) + \frac{1}{2} (0.01)^2 \cdot 2 = 4 + 0.04 + 0.0001 = 4.0401 \]

but even if I stopped only at first order, I'd have gotten a pretty decent approximation.

What Taylor series does for \( f(x) = x^2 \) is simple:

\[ (x+\Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2 \]

\[ \Rightarrow \text{No magic!} \]
Differential equations

A diff. eq. is an equation where the unknown is a function, where derivatives appear in the equation itself.

Example: \( f'(t) = \beta \cdot f(t) \), \( \beta \in \mathbb{R} \).

This is an ODE where the unknown is a function of a single real variable.

ODEs describe evolution, whereas the solution will be in a moment.

Separable ODE:

\[
\frac{df(t)}{dt} = \beta \cdot f(t)
\]

\[
\frac{df(t)}{f(t)} = \beta \cdot dx
\]

\[
\int \frac{df(t)}{f(t)} = \beta \int dx = \beta t + C
\]

\[
\log f(t) = \beta t + C
\]

\[
f(t) = C e^{\beta t}
\]

\[
f(t) = \int_{t_0}^{t} e^{\beta \tau} d\tau
\]
ODEs describe evolution:

Suppose I know what is the value of the function at \( t_0 \) (could be zero).

\[ f(t_0) = f_0 \implies \text{I can compute } f(t_0 + \Delta t) \]

\((i)\) \[ f(t_0 + \Delta t) = f(t_0) + \Delta t \cdot f'(t_0) = f_0 + \Delta t \cdot f'(t_0) \]

\((ii)\) Use the diff. eq.: \[ f'(t_0) = \beta \cdot f(t_0) = \beta f_0 \]

\[ f(t_0 + \Delta t) = f_0 + f_0 \cdot (\beta \Delta t) + O(\Delta t)^2 \]

\( \implies \) we know the value of the function at \( t_0 + \Delta t \) for very small \( \Delta t \).

\((iii)\) At this point \( t_0 + \Delta t \) I start the process again to obtain the value of the function in a further away point.

\[ f(A) \]

\[ f_0 \]

\[ x_0 \]

---

Graphical representation
Will this process provide us the full solution? \( \rightarrow \) YES!

Let us check for this particular case:

\( f'(t) = \beta t \).

Taylor series: 
\[
f(t_0 + t) = f(t_0) + t \cdot f'(t_0) + \frac{f''(t_0)}{2} t^2 + \frac{f'''(t_0)}{3!} t^3 + \ldots + \frac{f^{(n)}(t_0)}{n!} t^n + \ldots
\]

We have:

\[
f'(t_0) = \beta f(t_0) = \beta f_0
\]

\[
f''(t_0) = \beta f'(t_0) = \beta^2 f_0
\]

\[
f'''(t_0) = \beta f''(t_0) = \beta^3 f_0
\]

\[
\vdots
\]

\[
f^{(n)}(t_0) = \beta^n f_0
\]

Substitution into Taylor series yields:

\[
f(t_0 + t) = f_0 + (t \beta) f_0 + \frac{(t \beta)^2}{2} f_0 + \frac{(t \beta)^3}{3!} f_0 + \ldots + \frac{(t \beta)^n}{n!} f_0 + \ldots = f_0 \sum_{n=0}^\infty \frac{(t \beta)^n}{n!} = f_0 e^{\beta t}
\]

\[
f(t_0 + t) = f_0 e^{\beta t} \Rightarrow \text{this iterative procedure gives the exact value determines the full evolution of the system.}
\]
Equation of motion in some specific cases:

Example 1: Dimension, constant force:

(a) \( F = 0 \)

\[ m \cdot \ddot{x}(t) = F = 0. \]

\[ m \cdot \frac{d^2v}{dt^2} = 0 \Rightarrow \dot{v} = v = \text{constant} \]

\[ \frac{dx}{dt} = v_0 \Rightarrow x(t) = v_0t + x_0. \]

\( \Rightarrow \) Newton's first law!

In the absence of external forces objects maintain a constant velocity.

(b) \( F = F_0 \neq 0 \)

\[ m \cdot \frac{d^2\dot{v}}{dt^2} = F_0 \Rightarrow \ddot{v}(t) = \frac{F_0}{m} t \]

\[ \frac{dx}{dt} = v(t) = v_0 + \frac{F_0}{m} t \rightarrow x(t) = x_0 + v_0t + \frac{1}{2} \frac{F_0}{m} t^2 \]

uniformly accelerated motion.
2 Dimensions: Need to use vector components!

Suppose there is a constant force $\vec{F} = \text{constant}(t)$.

(§) Choose a reference frame $(x, y)$ so that $\vec{F} = F_x \hat{i} + F_y \hat{j}$.

Then if the trajectory of the particle is described by

$$\vec{r}(t) = \vec{x}(t) = x(t) \hat{i} + y(t) \hat{j}$$

so that

$$\vec{r}(t) = \dot{x}(t) \hat{i} + \dot{y}(t) \hat{j}$$

- velocity

$$\vec{v}(t) = \ddot{x}(t) \hat{i} + \ddot{y}(t) \hat{j}$$

- acceleration

then vector equation of motion

$$m \vec{a}(t) = \vec{F}$$

can be decomposed into

$$\begin{cases} m \ddot{x} = F_x \\ m \ddot{y} = F_y \end{cases}$$

two equations! (for one particle)

Example:

- $\vec{x} = 0 \Rightarrow x(t) = x_0 + V_{ox} t$

- $\vec{y} = -g \Rightarrow y(t) = y_0 + V_{oy} t - \frac{1}{2} gt^2$

Motion of a projectile:

- $F_x = 0$

- $F_y = -mg$
Example 3: Drag Force: \[ \mathbf{F} = -8 \mathbf{v} \]

The motion corresponds to the motion in a viscous medium.

\[ m \frac{d\mathbf{v}}{dt} = -8 \mathbf{v}, \]

Since \( \mathbf{F} \) is always parallel to \( \mathbf{v} \), the system is effectively one-dimensional (can always direct the coordinate axis along \( \mathbf{v} \)).

\[ m \frac{d\mathbf{v}}{dt} = -8 \mathbf{v} \]

can be solved by separation of variables.

\[ V(t) = V_0 e^{-\frac{8}{m} (t-t_0)} \]

\[ \Rightarrow \text{particle slows down in time and stops at } \]

\[ t \gg \frac{m}{8}. \]

How about \( x(t) \)?

\[ x(t) = \mathbf{v} = V_0 e^{-\frac{8}{m} (t-t_0)} \Rightarrow \frac{dx}{dt} = V_0 e^{-\frac{8}{m} (t-t_0)} \]

\[ x(t) - x(t_0) = V_0 \int_{t_0}^{t} e^{-\frac{8}{m} (t'-t_0)} dt' = \]

\[ = -\frac{mV_0}{8} e^{-\frac{8}{m} (t-t_0)} \bigg|_{t_0}^{t} = \frac{mV_0}{8} \left( 1 - e^{-\frac{8}{m} (t-t_0)} \right). \]
Bottom line: \[ x(t) = x_0 + \frac{mV_0}{g} \left[ 1 - e^{-\frac{g}{m} (t-t_0)} \right] \]

Note \[ x\left(t \gg \frac{m}{g}\right) \approx x_0 + \frac{mV_0}{g} \] - distance it travels.

Example 4: Time-dependent force:

1D example \[ F(t) = F_0 \sin(wt) \] with

\[ x(t=0) = 0 \]
\[ v(t=0) = 0 \]

Equation of motion reads: \[ \frac{dv}{dt} = \frac{F_0}{m} \sin wt \]

\[ \int_0^t \frac{dv}{dt} \, dt = \frac{F_0}{m} \int_0^t \sin wt \, dt' \]

\[ v(t) - v(t=0) = \frac{F_0}{m} \cos (wt) \bigg|_0^t \]

\[ v(t) = \frac{F_0}{m} \left[ 1 - \cos (wt) \right] \]

So \[ \frac{dx}{dt} = \frac{F_0}{mw} \left[ 1 - \cos (wt) \right] \], that can be integrated to

\[ x(t) = \frac{F_0}{mw} \left[ t - \frac{1}{w} \sin wt \right] \]. The particle has a net motion!