Outline of 24:

- Concepts of a scalar and a vector
- Coordinate transformations in 2D and 3D
- Rotation matrices and their properties
Basic math concepts: linear algebra

* Concept of a **Scalar**.

Quantities that are invariant under coordinate transformations are termed scalars.

**Example:** The mass of the particle is a scalar.

\[ M(x, y) \]

\[ O \]

\[ J (x_0, y_0) \quad M(x_0, y_0) \quad \Rightarrow \quad M(x_2, y_2) = M(x_0, y_0) \]

\[ O (x_1, y_1) \]

\[ \Rightarrow \text{mass of the particle} \]

\[ \text{is a scalar.} \]

Other examples: temperature, speed, etc.

**Note:**

On the other hand: direction of motion / direction of the force /

*cannot be specified in such a manner.*

These are vectors.

\[ x' = x_1 \cos \theta + x_2 \sin \theta = x_1 \cos \theta + x_2 \cos (\frac{\pi}{2} - \theta) \]
The position of $P$ can be represented in two coordinate systems, one rotated from the other.

Initial system $P(x_1, x_2, x_3)$, Final system $P(x'_1, x'_2, x'_3)$

Under rotation, coordinates transform as

$$x'_1 = Oa + ab + bc = x_1 \cos \theta + x_2 \sin \theta$$

$$= x_1 \cos \theta + x_2 \cos \left(\frac{\pi}{2} - \theta\right)$$

$$x'_2 = Od - de = Od - 0f =$$

$$= -x_1 \sin \theta + x_2 \cos \theta = x_1 \cos \left(\frac{\pi}{2} + \theta\right) + x_2 \cos \theta$$
Now we introduce the following notation: we write $	heta$ as the angle between $x_i$-axis and $x_i'$-axis.

\[
\theta = \Delta x_i', x_i = (x_i', x_i).
\]

In general we write $\Delta x_i', x_j = (x_i', x_j)$.

Furthermore, we define a set of numbers

\[\lambda_{ij} = \cos (x_i', x_j) \Rightarrow \text{called direction cosine of the } x_i' \text{ axis relative to the } x_j \text{ axis.}\]

For our 2D rotation we have

\[
\lambda_{11} = \cos (x_1', x_1) = \cos \theta \\
\lambda_{12} = \cos (x_1', x_2) = \cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta \\
\lambda_{21} = \cos (x_2', x_1) = \cos \left(\frac{\pi}{2} + \theta\right) = -\sin \theta \\
\lambda_{22} = \cos (x_2', x_2) = \cos \theta
\]

\[\Rightarrow \text{equations of transformation now become}\]

\[
x_1' = x_1 \cos (x_1', x_1) + x_2 \cos (x_1', x_2) = \lambda_{11} x_1 + \lambda_{12} x_2 \\
x_2' = x_1 \cos (x_2', x_1) + x_2 \cos (x_2', x_2) = \lambda_{21} x_1 + \lambda_{22} x_2 \\
x_3' = x_3
\]
In general, in 3 dimensions we have

\[
\begin{align*}
  x_1' &= \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3 \\
  x_2' &= \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3 \\
  x_3' &= \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3
\end{align*}
\]

or, in summation notation:

\[
x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j, \quad i = 1, 2, 3.
\]

The inverse transformation is

\[
x_i = \sum_{j=1}^{3} \lambda_{ji} x_j', \quad i = 1, 2, 3.
\]

Finally, if one writes a coordinate of \( P \) as a column/row

\[
\overrightarrow{P} (x_1, y_2, z_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

or \( \overrightarrow{P} (x_1, y_2, z_3) = (x_1, x_2, x_3) \),

then the matrix coordinate transformation will be read as:

\[
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
Properties of rotation matrices:

Geometric identities:

Line $l$ is given by $(a, b, d)$

Identity 1: $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

Identity 2: $\cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'$
Homework (will be assigned on Thursday):

Use these identities show that

\[ \sum_{j=1}^{3} \lambda_{ij} \lambda_{kj} = 0, \quad k \neq j. \]

(i.e. \(\lambda_{11}.\lambda_{21} + \lambda_{12}.\lambda_{22} + \lambda_{13}.\lambda_{23} = C_3 \bar{\lambda}_2 = 0\))

\[ \sum_{j=1}^{3} \lambda_{ij} \lambda_{kj} = 1, \quad i = k \]

(i.e. \(\lambda_{11}^2 + \lambda_{22}^2 + \lambda_{33}^2 = 1\)).

One may combine these relations into one

\[ \sum_{j=1}^{3} \lambda_{ij} \lambda_{kj} = \delta_{ik}, \]

where \(\delta_{ik} = 0\) if \(i \neq k\) is called Kronecker delta symbol.

See these facts:

The validity of this eq. depends on the coordinate axes in each of the systems being mutually perpendicular.

\[ \Rightarrow \text{ orthogonal systems.} \]
Returning to our matrix notation:

\[
\begin{pmatrix}
  x_1' \\ x_2' \\ x_3'
\end{pmatrix} = \Lambda
\begin{pmatrix}
  x_1 \\ x_2 \\ x_3
\end{pmatrix}, \quad \text{where} \quad (\Lambda)_{ij} = \lambda_{ij}
\]

A transposed matrix is a matrix derived from an original matrix by interchange of rows and columns.

\[
(\Lambda^T)_{ij} = (\Lambda)_{ji} \quad \text{or} \quad \lambda^T_{ij} = \lambda_{ji}.
\]

We have that \((\Lambda^T)^T = \Lambda\).

An identity matrix is when multiplied by another matrix, leaves the latter unaffected.

\[
\Lambda \cdot \Lambda = \Lambda, \quad \Lambda \cdot I = \Lambda.
\]

\[
I \cdot \Lambda = \begin{pmatrix}
  1 & 0 \\ 0 & 1
\end{pmatrix} \begin{pmatrix}
  \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22}
\end{pmatrix} = \begin{pmatrix}
  \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22}
\end{pmatrix} = \Lambda.
\]
Orthogonal systems and their rotation matrices:

Consider orthogonal matrix $\mathbf{A}$ in 2D:

$$
\mathbf{A} = \begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{pmatrix},
$$

then

$$
\mathbf{A} \cdot \mathbf{A}^T = \begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{pmatrix} \cdot \begin{pmatrix}
\lambda_{11} & \lambda_{21} \\
\lambda_{12} & \lambda_{22}
\end{pmatrix} = \begin{pmatrix}
\lambda_{11}^2 + \lambda_{12}^2 & \lambda_{11} \lambda_{21} + \lambda_{12} \lambda_{22} \\
\lambda_{21} \lambda_{11} + \lambda_{22} \lambda_{12} & \lambda_{21}^2 + \lambda_{22}^2
\end{pmatrix}.
$$

Using the orthogonality relation we find

$$
\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2 + \lambda_{22}^2 = 1
$$

$$
\lambda_{11} \lambda_{21} + \lambda_{21} \lambda_{12} = \lambda_{11} \lambda_{21} + \lambda_{12} \lambda_{22} = 0
$$

$$
\Rightarrow \mathbf{A} \cdot \mathbf{A}^T = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \mathbf{I}.
$$

On the other hand, the inverse matrix is defined as:

$$
\mathbf{A}^{-1} = \mathbf{I} \rightarrow \text{identity matrix}
$$

Comparing these equations we see

$$
\mathbf{A}^T = \mathbf{A}^{-1}
$$

for orthogonal matrices.
Hartik algebra:

1. Matrix multiplication is not commutative in general
   \[ \Lambda_1 \cdot \Lambda_2 \neq \Lambda_2 \cdot \Lambda_1. \]

* The special case of multiplication of \( \Lambda \) and \( \Lambda^{-1} \) is commutative
   \[ \Lambda \cdot \Lambda^{-1} = \Lambda^{-1} \cdot \Lambda = \mathbb{I}. \]

* The identity matrix commutes: \( \Lambda \cdot \mathbb{I} = \mathbb{I} \cdot \Lambda = \mathbb{I} \).

2. Associativity:
   \[ (A \cdot B) \cdot C = A \cdot (B \cdot C). \]

3. Addition of matrices: \( \Lambda = \Lambda_1 + \Lambda_2 \) means
   \[ (\Lambda)_{ij} = (\Lambda_1)_{ij} + (\Lambda_2)_{ij}. \]
   - Defined if \( \Lambda_1 \) and \( \Lambda_2 \) have the same dimension.

Definition of a scalar and a vector:

Coordinate transform: \( x_i' = \sum_{j=1}^{3} l_{ij} x_j \) with \( \sum_{j=1}^{3} l_{ij} x_j = \delta_{ik} \).

a) If quantity \( \phi \) is unaffected, i.e., \( \phi(x_1, x_2, x_3) = \phi(x_1', x_2', x_3') \), then it is called a scalar.

b) If \( \mathbf{A} = (A_1, A_2, A_3) \) transforms as \( A_i' = \sum_{j=1}^{3} l_{ij} A_j \), then \( \mathbf{A} \) is called a vector.