Outline:

* Gauss' theorem
* Green's theorem
* Stokes' theorem
* Oersted's and Faraday's Laws
* Potential theory
Let \( \vec{V} = \vec{V}(r) \) - is a continuous vector and its first derivatives are also continuous over the simply connected region.

**Gauss's theorem:**

\[
\iiint_{\Omega} \vec{V} \cdot d\vec{S} = \iiint_{V} \nabla \cdot \vec{V} \, dV
\]

or

\[
\iint_{\partial V} \vec{V} \cdot d\vec{S} = \int_{V} \nabla \cdot \vec{V} \, dV , \quad \nabla \cdot \vec{V} = \text{div} \vec{V}.
\]

**Green's theorem:**

If \( u \) and \( \nabla \) are two scalar functions, we have the identities:

1. \( \nabla \cdot (u \nabla \nabla) = u \nabla \cdot \nabla \nabla + (\nabla u) \cdot (\nabla \nabla) \) \hspace{1cm} (1)

2. \( \nabla \cdot (\nabla \nabla u) = \nabla \cdot \nabla \nabla u + (\nabla \nabla) \cdot (\nabla u) \) \hspace{1cm} (2)

Subtracting (2) from (1) and integrating over a volume (where \( u, \nabla \) and their derivatives are assumed to be continuous), we obtain:
\[
\iiint_{V_t} (\mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}) \, d\tau = \int_{\partial V_t} (u \nabla v) - (v \nabla u) \cdot d\mathbf{\partial} \tag{3}
\]

Let us concentrate on the RHS and use Gauss' Theorem:

\[
\iiint_{V_o} \nabla \cdot \left[ (u \nabla v) - (v \nabla u) \right] \, d\tau =
\]

\[
= \iint_{\partial V_o} (u \nabla v - v \nabla u) \cdot d\mathbf{\partial}.
\]

Therefore Eq. (3) reduces to

\[
\iiint_{V_t} (\mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}) \, d\tau = \iint_{\partial V_o} (u \nabla v - v \nabla u) \cdot d\mathbf{\partial} \quad (Ov=5)
\]

This is called Green's theorem.

An alternate form of Green's theorem can be obtained from Eq. (1) alone:

\[
\iint_{\partial V} u \nabla v \cdot d\mathbf{\partial} = \iiint_{V_t} u \nabla v \, d\tau + \iiint_{V_t} \mathbf{v} \cdot \nabla d\mathbf{\partial}.
\]
Alternate forms of Gauss' Theorem:

Suppose our vector $\vec{V}$ has the following form:

$$\vec{V}(x, y, z) = V(x, y, z) \cdot \vec{a},$$

where $\vec{a}$ is a vector with constant magnitude and constant but arbitrary direction.

Using this $\vec{V}$ in Gauss' theorem one obtains:

$$\vec{a} \cdot \iiint_V \vec{V} \, d\tau = \iiint_V \nabla \cdot \vec{V} \, d\tau = \vec{a} \cdot \iiint_{\text{Vol}} \nabla \vec{V} \, d\tau$$

This may be rewritten as

$$\vec{a} \cdot \left[ \iiint_V \vec{V} \, d\tau - \iiint_{\text{Vol}} \nabla \vec{V} \, d\tau \right] = 0$$

Since $|\vec{a}| \neq 0$ and its direction is arbitrary, we have

$$\iiint_V \vec{V} \, d\tau = \iiint_{\text{Vol}} \nabla \vec{V} \, d\tau.$$ 

Similarly, using $\vec{V} = \vec{a} \times \vec{P}$, where $\vec{a}$ is a constant vector, we may show

$$\iiint_V d\vec{a} \times \vec{P} = \iiint_{\text{Vol}} \nabla \times \vec{P} \, d\tau.$$
Stokes' Theorem:

For a given \( \vec{V} = \vec{V}(x,y) \)

\[
\oint_{\partial S} \vec{V} \cdot d\vec{L} = \iint_S \nabla \times \vec{V} \cdot d\vec{S}
\]

where \( \partial S \) is the contour of \( S \).

Alternative Form:

As with Gauss' Theorem, other relations between surface and line integrals are possible:

\[
\oint_S (d\vec{S} \times \nabla) \cdot \vec{P} = \oint_{\partial S} d\vec{L} \times \vec{P}
\]

Eq. (4) is readily verified by taking \( \vec{V} = \vec{a} \cdot \nabla \Phi \) in the Stokes' theorem:

\[
\oint_S (\nabla \times (\vec{a} \cdot \nabla \Phi)) \cdot d\vec{S} = -\oint_{\partial S} \vec{a} \times \nabla \Phi \cdot d\vec{L}
\]

\[
= -\vec{a} \cdot \int_S \nabla \Phi \times d\vec{S}
\]
For the line integral we have
\[ \oint_C \vec{a} \cdot d\vec{e} = \vec{a} \cdot \oint_C \vec{q} \, d\vec{e}. \]
Therefore from Stokes' theorem we obtain
\[ \vec{a} \cdot \left( \oint_C \vec{q} \, d\vec{e} + \oint_S \vec{\nabla} \times \vec{u} \times d\vec{S} \right) = 0. \]
and this is true for arbitrary \( \vec{a} \) (with arbitrary direction) \( \Rightarrow \)
\[ \oint_C \vec{q} \, d\vec{e} + \oint_S \vec{\nabla} \times \vec{q} \times d\vec{S} = 0. \]
Eq. (5) can be obtained in a similar way by using \( \vec{E} = \vec{a} \times \vec{F} \), in which \( \vec{a} \) is again a constant vector.

Oersted Law: consider a magnetic field, \( \vec{H} \), generated by a long wire that carries a stationary current \( I \). Then from Maxwell's wire law \( \vec{\nabla} \times \vec{H} = \vec{J} \) - current density. Therefore
\[ \vec{J} = \oint_S \vec{E} \cdot d\vec{S} = \oint_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \oint_S \vec{H} \cdot d\vec{F}. \]
- called Oersted law.
Similarly, Maxwell's equation for \( \nabla \times \vec{E} = -\frac{dB}{dt} \), yields:

\[
\int \vec{E} \cdot d\vec{r} = \int (\nabla \times \vec{E}) \cdot d\vec{S} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S} = -\frac{d\phi}{dt}.
\]

This is called Faraday's law.

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**Potential Theory:**

A force \( \vec{F} \) is called conservative if \( \vec{F} = -\nabla \psi \), where \( \psi(\vec{r}) \) is a scalar potential.

Let's establish two equivalent relations:

\[
\vec{F} = -\nabla \psi
\]

\[
\nabla \times \vec{F} = 0
\]

\[
\oint \vec{F} \cdot d\vec{r} = 0
\]

Equivalent formulations of a conservative force.

Work done by force:

\[
\text{Work done by force} = \int_A^B \vec{F} \cdot d\vec{r} = -\int_A^B \nabla \psi \cdot d\vec{r} = \psi(A) - \psi(B)
\]
\[ \Rightarrow \quad \mathbf{F} \cdot d\mathbf{r} = -d\psi = -\nabla \psi \cdot d\mathbf{r} \]

This may be rewritten

\[ (\mathbf{F} + \nabla \psi) \cdot d\mathbf{r} = 0 \]

and since \( d\mathbf{r} \) is arbitrary, we have \( \mathbf{F} = -\nabla \psi \) follows from here directly.

From Stokes' theorem:

\[ 0 = \oint \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \mathbf{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} \text{ should be true for any surface} \]

\[ \Rightarrow \quad \nabla \times \mathbf{F} = 0! \]