Gauge Theories and Topological Order
In this chapter: 2+1d systems, phase transition with no local order parameter.

2+1d Ising Model and Duality

\[ H = -J \sum_{ij} \sigma_i^z \sigma_j^z - H \sum_\sigma \sigma^x \]

For \( H \ll J \): FM phase
  
  (almost) degenerate GS

Excitations:

- DW: Closed loops, not point-like objects anymore!
  - "String-like" excitations
  - Theory of such closed strings = binary version of electromagnetism.

DW Operator:
Let $T$'s live on the links of the dual lattice (since DW naturally live on the dual lattice).

As in Id.:

\[ \tau_{ij}^x = \sigma_i^x \sigma_j^x \]

counts DW between $i$ and $j$.

However: $N$ sites, $2N$ links $\Rightarrow$ $2N$ $T$'s!

we need $N$ constraints.

\[ \tau_1 \tau_{12} \tau_{13} \tau_{14} \tau_{15} = (\sigma_1^x \sigma_2^x) \sigma_3^x \sigma_4^x \sigma_5^x \sigma_7^x \sigma_9^x \sigma_8^x \sigma_6^x \sigma_5^x \sigma_8^x \sigma_7^x \sigma_9^x \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^x \sigma_6^x \]

$\Rightarrow$ local constraint

We write:

\[ \prod_{i=1}^{4} \tau_{i,i+1} = 1 \]

\[ \forall \text{ sites of the dual lattice} \]

sets of 4 links that emerge from site $i$ on dual lattice.

Spin Flip operator:

$\vec{\sigma}_i^x$ is a group operation that flips the value of DW operator $T_i^x$ on all bonds surrounding the lattice site $i$.

$\Rightarrow \sigma_i^x$ creates loop of DW: flips the value of DW operator $T_i^x$ on all bonds surrounding the lattice site $i$.

\[ \sigma_i^x = \tau_{12} \tau_{13} \tau_{14} \tau_{15} = \prod \tau_i^x \]

$\uparrow$ plaquette of the dual lattice
\[ H = -J \sum_{\text{links}} \tau^x_{ij} - K \sum_{\text{loops} \in \mathcal{L}} \tau^z_{ij} \]

\[ \tau^x \text{ live on links of square lattice} \]

\[ \prod_{i} \tau^x = 1 \text{ for all } i \]

Gauge invariance: \( H \) invariant under local \( \mathbb{Z}_2 \) gauge transformation:

\[ \tau^z_{ij} \rightarrow \epsilon_i \tau^z_{ij} \epsilon_j \text{ with } \epsilon_i = \pm 1 \in \mathbb{Z}_2 \]

Clearly, the 2+1d Ising model isn't self-dual! As in 1d, the mapping isn't 1 to 1 (no symmetry breaking and 65 degeneracy for the Gauge theory at small \( h \)).

\[ \Rightarrow \] Constraints: \( V_i: \prod \tau^x = 1 \]

\[ \text{commute: constant on 0 or 2 links, } (-1)^c = 1 \]

In the following we will forget about the Ising model, and treat the dual gauge theory as "fundamental."
H is invariant for any such local transformation. This is sometimes called "local symmetry," but this is really a Gauge redundancy of the theory. This transformation doesn't really change states like a spin flip symmetry in the Ising model.

\[ G_i |\psi\rangle = |\psi\rangle \] states invariant under Gauge transformation

\[ L \] generates Gauge transformation with \( E_i = -1 \)
\[ E_{ij} = 1 \quad \forall j \neq i \]

\[ G_i = \prod_{+i} r^{-x} \] (Plays the sign of all \( r^\tau \)'s emanating from \( i \))

\( = \) constraints ensures \( G_i = 1 \): keeps only physical states.

\( \mathbb{Z}_2 \) Electromagnetism:

\[ r^z = e^{i\pi a_{id}} \quad a_{id} = 0,1 \]

\[ \prod r^z = e^{i\pi \sum a_{id}} = e^{i\pi \theta} \] " Flux through plaquette

let \( r^x = e^{i\pi \theta} \) so \( H \sim (-1)^e + g(-1)^e \)

Constraints:

\[ \prod r^{-x} = e^{i\pi \nabla \cdot e} = 1 \quad \Rightarrow \quad \nabla \cdot e = 0 \mod 2 \]

where \( \sum e_{id} = e_{\pi + \tilde{y}/2} + e_{\pi - \tilde{y}/2} + e_{\tilde{x} + \tilde{z}/2} + e_{\tilde{x} - \tilde{z}/2} \)

\( \nabla \cdot e \) can be flipped to \( 0 \) since \( e \equiv -e \)

Electric Field = DW Forms closed loops
Phase diagram of the Ising Gauge Theory

- **Confined Phase**: \( g \ll 1 \): For \( g = 0 \),
  \[ H = -J \sum_{ij} T_{ij}^x \]
  \[ \Rightarrow T_{ij}^x = 1 \] on all links, satisfies constraint.
  This corresponds to \( e = 0 \) everywhere.

For \( g \) small, there will be some links with non-zero electric fields. To satisfy the constraints, the field lines have to form closed loops. For small \( g \), we expect these loops to be small and dilute. As we increase \( g \), these loops proliferate = condense.

- Confinement of test changes: electric lines are confined for \( g \ll 1 \). To see this, insert two “test changes” at sites \( i \) and \( i + \mathbf{p} \), and ask how much energy it costs to pull these changes apart.

Changes: \( T_{i} T_{ij}^x = -1 \) (odd number of \( T_{ij}^x = -1 \) emanate from this site and have to connect to the other test change)

Each \( T_{ij}^x = -1 \) links costs energy \( 2J \): pick shortest path:

\[ \Delta E(\mathbf{p}) = 2J \mathbf{p} \]

- In this phase, the electric field is well defined (\( \approx 0 \)) while the magnetic field fluctuates wildly.

- Remark: It’s important to perform this diagnosis in a pure gauge theory (without matter, additional changes)
Deconfined Phase: $g \gg 1, \quad H = -gJ \sum_{\Box_i} \tau_i^z \tau_j^z \tau_k^z \tau_l^z$

$\text{GS: } \prod_{\Box_i} \tau_i^z = +1 \quad \text{for all } \Box_i$

(Gapped) Excitation: Flip a given plaquette to $\prod_{\Box_i} \tau_i^z = -1$

energy cost: $\Delta = 2gJ$. To create such an excitation, we actually need to flip flips along a “string”

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & B_0 = -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

B = $\prod_{\Box_i} \tau_i^z$

Apply $\tau^x$ on $-1$ links

This excitation comes with a string attached.

GS wavefunction: Let’s first work in the “$B = \tau^z$” basis.

Naively: $|\Psi_0^+\rangle = \bigotimes_{i,j} |\tau_i^z = +1\rangle$ But not Gauge invariant!

$G_i = \prod_{-\tau^x} \tau_i^z \quad \tau^z \rightarrow -\tau^z$

$\rightarrow |\Psi_0^+\rangle = \prod_i \left( 1 + G_i \right) |\Psi_0^+\rangle$ now $G_i(\text{GS}) = G_i$

projection onto $G_i = +1$

and $H |\Psi_0^+\rangle = E_0 |\Psi_0^+\rangle$ since $[H, G_i] = 0$

$H |\Psi_0^-\rangle = E_0 |\Psi_0^-\rangle$
Deconfinement of test changes:

\[ \text{insert two test changes } x_0, x_0 + p \]

\[ |G_0 > = \left( 1 - G \frac{x_0^2}{2} \right) \left( 1 - G \frac{x_0^2 + p_0^2}{2} \right) \prod_{\delta} \left( 1 + G_{\delta} \right) |\Psi_0 > \]

\[ H (G_0) = E_0 |G_0 > \Rightarrow \Delta E(p) = 0 \quad \text{in this limit!} \]

Charges are deconfined (\( \Delta E(p) \rightarrow \infty \))

In \( E = \tau^x \) basis:

\[ |r^z = +1 > = \frac{|r^x = +1 > + |r^x = -1 >}{2} \]

\[ = \cos \frac{1}{e = +1} + \sin \frac{1}{e = 0} \]

\[ \prod_{\delta} \tau^z |G_0 > = |G_0 > \quad \text{as } \prod \tau^z \text{ creates a loop } \]

- deconfined phase: strings are "cheap" and fluctuating.

\[ \text{Topological Order} \]

Let's consider the deconfined phase.

Excitation = gapped magnetic flux excitation = "vison"

Carries \( \mathbb{Z}_2 \) flux of -1 = \( \pi - \text{flux} \)

As explained above, these excitations come with a string.
Acting with $T^x$ along the string (red links) creates two visions ($-1 \mathbb{Z}_2$ fluxes).

The string is not measurable by any local measurement; all the plaquettes along the string have no fluxes, $B_0 = +1$.

**Topological GS degeneracy:** $K = \#$ non contractible paths

\[
\begin{array}{c}
K = 2 \\
K = 1 \\
K = 1
\end{array}
\]

GS degeneracy $= 2^K$

**Proof:** Consider a cylinder and $g = \infty$ limit (for now).

$X = \prod T^x_e$: creates visions at $\pm \infty$.

$Z = \prod p^z_e$: takes electric charge around the cylinder.

gauge invariant operators
Take a ground state \( |GS\rangle \). Then \( X|GS\rangle \) also \( GS \) since all plaquettes have 0 \( Z_2 \) fluxes through them. \( [H, X] = 0 \) also \( [H, Z] = 0 \).

But \( \{Z, X\} = 0 \) (share one link, and \( T_x, T^3 = 0 \)) which should be represented on the GS:

\[
|GS\rangle = X|GS\rangle \\
Z|GS\rangle = ZX|GS\rangle = -XZ|GS\rangle = -X|GS\rangle
\]

- gauge invariant = physical

\[
\text{two-fold degeneracy!}
\]

\[
Z|GS\rangle = Z|GS\rangle \quad \Rightarrow \quad Z \text{ measure, the } Z_2 \pi \text{ flux created by } X \text{ in } |GS\rangle
\]

More formally, since \( H \) commutes with \( Z \) and \( X \), the anti-commutation \( \{Z, X\} \) should be represented on the GS

\[
Z^2 = X^2 = 1 \quad \Rightarrow \quad \text{implies degeneracy}
\]

Note: Precise contours do not matter. \( C \) can be deformed by acting by \( G_i \).

\[
8<\infty: \quad [H, X] = 0 \quad \text{But} \quad [H, Z] \neq 0 \text{ now.}
\]

\[
\begin{align*}
\text{treated as } -J \sum T_x \text{ using perturbation theory.}
\end{align*}
\]

|GS\rangle \text{ and } X|GS\rangle \text{ now related by matrix element}

\[
H_{eff} = \begin{pmatrix} E_0 & \Gamma \\ \Gamma & -E_0 \end{pmatrix}, \text{ actual ground states are superpositions of}
\]

|GS\rangle \text{ and } X|GS\rangle

\[
\Gamma \approx J \left( \frac{J}{2J} \right)^L \\
\text{undo string of flipped}
\]

\[
\tau^3 = -1.
\]

\[
\text{at any step, unhappy plaquette energy cost: } 2J g \text{ (large)}
\]
No local operator can tell the difference between $|GS\rangle$ and $|IGS\rangle$. The flux can only be measured by taking an electric change all around the cylinder.

IV. Tonic Code

In the $\mathbb{Z}_2$ gauge theory, the physical objects are electric loops (strings), and the Hilbert space doesn't really have a tensor product structure because of the gauge constraint. Can we have this structure emerge in a physical spin model?

$\implies$ implement constraint "dynamically" (high energy cost for violating it)

$$H_{Tc} = - J_m \sum_0 B_0 - J_e \sum A_+ \tag{Kitaev}$$

with $B_0 = \prod_{i \in 0} \tau_i^z$, $A_+ = \prod_{i \in +} \tau_i^x$

$D$ = plaquette = $P$

$+ =$ star term = $S$

no constraint in this model

(But $J_e \to 0$ enforces previous gauge constraint)

Exact solution: sum of commuting terms:

$$[A_5, B_p] = 0$$
$$[A_5, A_{5'}] = 0$$
$$[B_p, B_{p'}] = 0$$

$\forall s, s', p, p'$

$GS$: $A_5 = +1$, $B_p = +1$ ($A_5 = +1$ emerges dynamically!)
\[ B_p = -1 : \text{vortex (magnetic) excitation} \]

\[ B_p |GS\rangle = +1 |GS\rangle \quad \forall p \Rightarrow |GS\rangle = \sum_{p=1}^{\infty} c_{p=1} \epsilon^{p} |p\rangle \text{ s.t. } \prod_{i=0}^{\infty} \tau_{i}^{t_{i}+1} \text{ i.e. } (\text{no flux}) \]

\[ \Rightarrow |GS\rangle = \text{superposition of vortex-free configurations} \]

Now think of \( \tau_{t_{i}+1} \) as links.

\[ A_{5} |GS\rangle = |GS\rangle \quad \forall s \]

On infinite plane, \( C_{p=1} = +1 \) up to normalization (as the \( A_{5} \) generate any configuration from \( |\tau_{t_{i}+1}\rangle \)).

On Torus:

\[ W(\tau_{t_{i}+1}) = \prod_{i \in \epsilon} \tau_{i}^{t_{i}_{+1}} \text{ ("Wilson" loops)} \]

Any \( A_{5} \) will intersect 0 or 2 edges of these loops. Hence cannot connect states with different values of \( \omega_{\epsilon_{1}}, \omega_{\epsilon_{2}} \).

\[ \Rightarrow 4 \text{ degenerate GS: } (\omega_{\epsilon_{1}}, \omega_{\epsilon_{2}}) = (\pm 1, \pm 1) \]

Excitations: two flavours: electric changes and magnetic vortices.

A team \quad B team
Electric path operator:

\[ W^{e_{i}}_{s_{i},s_{2}} = \prod_{i \in E_{s_{i},s_{2}}} \tau_{i} \]

Connects 2 stars

Clearly \([W^{e_{i}} B_{p}] = 0\) for all \(A_{s}^{e} \), except \(A_{s_{1}}, A_{s_{2}}\) (share only 1 link)

\(\langle \psi_{s_{1},s_{2}} \rangle = W^{e_{i}}_{s_{1},s_{2}} |G_{S}\rangle\)

Eigenstate with energy \(4J_{e}\)

\(\psi_{s_{1},s_{2}}\) is an eigenstate with energy \(4J_{e}\).

\(\psi_{s_{1},s_{2}}\) is an eigenstate with energy \(4J_{e}\).

Magnetic path operator:

\[ W^{m}_{\mathbb{P}_{i}} = \prod_{i \in \mathbb{E}_{\mathbb{P}_{i}}} \tau_{i}^{x} \]

Connects two plaquettes, \(\mathbb{E} = \mathbb{P}_{i}\) on dual lattice

Commutes with all \(A_{s}^{e}\), and almost all \(B_{p_{i}}^{e}\), almost all \(B_{p_{i}}^{e}\)

\(\langle \psi_{\mathbb{P}_{i},\mathbb{P}_{j}}^{m} \rangle = W^{m}_{\mathbb{P}_{i}} |G_{S}\rangle\)

Energy = \(4J_{m}\)

\(\psi_{\mathbb{P}_{i},\mathbb{P}_{j}}^{m}\) is an eigenstate with energy \(4J_{m}\).

\(\mathbb{P}_{i}\) is an eigenstate with energy \(4J_{m}\).

\(\mathbb{P}_{i}\) is an eigenstate with energy \(4J_{m}\).

Note: There's no phase transition in \(H_{Te}\)

(Commuting projector Hamiltonian)
Anyonic Statistics and Emergent Fermions:

Exchange identical particles, focus on statistical phase:

\[ \begin{align*}
& a \leftrightarrow b = R_{ab} & \text{indistinguishable particles: } a = b \\
& \text{Do this twice:} \\
& \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} & \begin{array}{c}
\text{a} \\
\circ \\
\text{b}
\end{array} \\
\end{align*} \]

in 3d: Full notation two exchanges do nothing \( \Rightarrow \) identity.

in 3d \( \rightarrow \) exchange can lead to \( \pm 1 \) eigenvalues (neutrinos in bosons) in 2d: Braid group more complicated \( \rightarrow \) Anyons.

Toric code: Clearly, \( e \) and \( m \) are bosons since path operators of the same path commute with each other.

We write:

\[ \begin{align*}
& \begin{array}{c}
\text{e} \\
\circ \\
\text{e} \\
\circ \\
\text{e}
\end{array} = \\
& \begin{array}{c}
\text{e} \\
\circ \\
\text{e} \\
\circ \\
\text{e}
\end{array} \\
& \begin{array}{c}
\text{m} \\
\circ \\
\text{m} \\
\circ \\
\text{m}
\end{array} = \\
& \begin{array}{c}
\text{m} \\
\circ \\
\text{m} \\
\circ \\
\text{m}
\end{array} \\
\end{align*} \]

However, they have some non-trivial mutual statistics:

\[ \begin{align*}
& \begin{array}{c}
\text{e} \\
\circ \\
\text{m}
\end{array} = \\
& \begin{array}{c}
\text{m} \\
\circ \\
\text{e}
\end{array} \\
\end{align*} \]

Mutual statistics of \( \pi \).
To see this, let's consider $|p_i>$, a state with magnetic vortex at $p_i$.

"Braiding operation": Take e charge around $m$:

$$|p_i> \rightarrow \prod_{i \in \mathcal{E}} \tau_{i}^{z} |p_i>$$

$\mathcal{E}$ contour surrounding $p_i$.

Now:

$$\prod_{i \in \mathcal{E}} \tau_{i}^{z} = \prod_{\text{inside } \mathcal{E}} B_p$$

("Stokes' Theorem!")

$$\oint_{\mathcal{E}} \mathbf{w} = \int_{\Omega} \mathbf{dw}$$

and $B_p |p_i> = -|p_i>$.

So $|p_i> \rightarrow -|p_i>$ under this braiding operation.

$\prod_{i} \tau_{i}^{z}$ isn't trivial if $\mathcal{E}$ encloses a magnetic vortex!

$\tau_{i}^{z}$ is a fermion.

This means that $E = e \times m$ is a **Beamsion**:

$\tau_{i}$ composite particle
The GS degeneracy can also be understood in terms of this non-trivial mutual statistics.

Note: create ee pair and wrap up of all of them around a cycle of the towns to annihilate them again, and same thing for mm around the other cycle: those operations anticommute.

VI. \( \mathbb{Z}_2 \) Gauge theory with “manner fields”

\( \tau_{ij} \): gauge “fields”, live on links

\( \sigma_i \): Ising matter fields, live on sites. (vertex)

\[
H = -g \sum_{\langle ij \rangle} \tau_{ij}^x - g^{-1} \sum_0 \tau_{ij}^z \tau_{jk}^z \tau_{kl}^z - \lambda \sum_i \sigma_i^x - \lambda \sum_i \sigma_i^z \tau_{ij}^x \sigma_j^z
\]

Gauge symmetry: \( \sigma_i^x \prod_{\partial \ell_i} \tau_{ij}^x = \prod_{\partial \ell_i} \tau_{ij}^x = \sigma_i^x \) for \( g \in \ell_i \) since \( A_i = \sigma_i^x \)

(\( \nabla \cdot e = p \))

\( \sigma = \mathbb{Z}_2 \) electric charge

\[
\begin{align*}
\text{if } & g \rightarrow 0, \quad H = -g^{-1} \sum_0 B_0 - \lambda \sum_+ A_+ \quad \text{since } A_+ = \sigma_i^x \\
\lambda \rightarrow 0 \quad & = H_{\text{Tr}}
\end{align*}
\]
\( g = 0, \lambda \neq 0: \) pure matter theory

\[ B_0 = +1, \forall \Box \]

(no flux condition)

Under gauge fixing: \( \tau_{ij}^2 = +1 \) on all links

\[ H_{G.F.} = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{ij} \sigma_i^z \sigma_j^z \]

has a symmetry breaking transition as a function of \( \lambda \). \( \lambda \) large: Higgs phase

\( \Rightarrow \) conventional symmetry breaking transition upon gauge fixing.

\( \lambda \) give dynamics to e changes. At the Higgs transition, e particles condense (\( \sigma \) gets expectation value)

leads to confinement of m (general Topological QFT result)

\( e, m \) have non trivial mutual statistics

\( \lambda = 0, g \neq 0: \) pure gauge theory

electric changes now cost \( \alpha \) energy \( \Rightarrow \) constraint \( \prod \tau_x = +1 \) on each star +

\[ H = H_{Z_2 \text{ gauge theory}}. \]

As \( g \) increases: confinement of e particles

m particles condense

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**Diagram:**

- Pure-Matter Theory
- Deconfined Phase
- \( Z_2 \) Top. Order
- Pure gauge Theory

**Diagrame Notes:**
- Higgs
- Change condensate
- Vortices confined
- \( \text{Deconfined Phase} \)
- Vortex Condensate
- \( \text{Charge} \) changes confined

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\[ g = 0, \lambda \neq 0: \text{pure matter theory} \]

\[ B_0 = +1, \forall \Box \] (no flux condition)
Higgs Phase and "Spontaneous Gauge Symmetry Breaking"

- **Xiao-Gang Wen** argument: gauge "symmetries" aren't actual symmetries, "do nothing" transformation. Two states related by gauge transformation are actually the same state. Can't be spontaneously broken.

- **Elitzur's theorem**: Gauge symmetries can't be spontaneously broken.

**Intuitively**: In a 2d classical Ising model, going from all $\uparrow$ to all $\downarrow$

requires a domain wall with extensive energy cost. In 1d: no extensive energy cost, entropy wins $\Rightarrow$ no FM phase in classical 1d Ising model

$\Rightarrow$ same argument breaks down for local gauge symmetries: different GS would be connected by local gauge transformations at no energy cost!

So what's going on in the Higgs phase?

\[ H = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \]

with $B_{ij} = r^2 r^2 r^2 r^2 = +1$ (Focus on $g \to 0$)

Gauge fixing: $T^z_\mu = 1$, then looks like spontaneous symmetry breaking?

Solution: $T^z_\mu = \sigma^z_\mu$ as $\lambda \to \infty$, $g \to 0$ (satisfies $B_{ij} = +1$, 4D)

The true eigenstates of $H$ can be obtained from:

\[ H_{\text{g-fixed}} = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \]

\[^{\text{gauge fixed Hamiltonian}}\]
By symmetrizing to make them gauge invariant:

\[ |y_n\rangle = \sum_G G |y_n\rangle \quad G = \text{gauge transformations} \]

In particular, the two ferromagnetic groundstates of \( H_{\text{g.b.}} \):

\[ |\sigma^z=+1, \tau^z=+1\rangle \quad \text{and} \quad |\sigma^z=-1, \tau^z=+1\rangle \]

are related by gauge transformations.

\[ \Rightarrow \] the Higgs mechanism looks like SSB for a particular choice of gauge, but the true GS is unique and gauge-invariant.

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### VII Detecting Topological Order using Entanglement

How can one detect deconfinement experimentally or even numerically?

- degeneracy on torus with no broken symmetry/order
- Braiding properties of anyonic excitations.

Say we have the wavefunction \( |\psi\rangle \) that is the GS of a topologically ordered \( H \). No torus, no excitation: How do we tell that it is topologically non-trivial?

\[ \Rightarrow \text{Entanglement Entropy} \]

\[ S_A = - \sum_{\rho_A} \rho_A \log \rho_A \quad \rho_A = \text{tr}_A \rho \quad \text{with} \quad \rho = |\psi\rangle \langle \psi| \]

\[ \rho_A = |\psi\rangle \langle \psi| \]

\[ \text{reduced density matrix} \]

Example: two qubits:

\[ |\psi\rangle = |\uparrow\rangle \quad \Rightarrow \quad \rho = |\uparrow\rangle \langle \uparrow| \]

\[ \rho_A = |\uparrow\rangle \langle \uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ S_A = 0 \quad \text{not entangled product state} \]
\[ |\psi\rangle = \frac{|1\rangle + |1\rangle}{\sqrt{2}} \Rightarrow p = \frac{1}{2} (|1\rangle \langle 1| + |1\rangle \langle 1| + |1\rangle \langle 1| + |1\rangle \langle 1|) \]

\[ p_A = \frac{1}{2} (|1\rangle \langle 1| + |1\rangle \langle 1|) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

Mixed state!

\[ S_A = - \sum_i p_i \log p_i = \log 2 \text{ with } \sum_i p_i = 1 \]

Properties:
- For a pure state: \( S_A = S_B \)
- Strong subadditivity:
  \[ S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \]

Many-body system:
- \( S_A \leq N_A \log 2 \)
- \( N_A \sim L_A^d \) spins \( \frac{1}{2} \) in 2d. “volume law”
- Gapped ground states:
  \( S_A \sim L_A \) area law
- CFT in 1+1d:
  \( S_A = \frac{c}{3} \log L_A \) violates (mildly) area law
- Low entanglement in quantum GS:
  Tensor networks and matrix product state techniques. DMRG etc.

Entanglement in Tonic code:
- \( S_A \sim \alpha L_A - \gamma \) \( \alpha \) universal, = \( \log 2 \) \( \gamma = \log 2 \) \( \mathbb{Z}_2 \) top. order

\[ |\psi\rangle = \sum_{\text{electric loops}} |\text{loop}\rangle \]

\( A \): compact simply connected.

Eigenvalues of \( p_A \) can be labelled by electric \((\tau^x)\) configurations at the boundary.

Loops that do not cross the boundary do not contribute to \( S_A \)
Naively $2^{LA}$ possibilities for $T^X = 1$ at boundary, all equiprobable in $|4\rangle$.

However: Non local constraint from (emergent) Gauss law: number of electric line crossing boundary is Even.

$= N = 2^{LA-1}$ possible configurations (last e line fixed) Gauss law gives us “one bit” of information.

$p_i = \frac{1}{N}$

$S_A = -\sum_{i=1}^{N} p_i \log p_i = \log N = LA \log 2 - \log 2$

This is a universal property of $Z_2$ topologically ordered states.

\[ \text{VII U(1) Gauge Theories (Very Brief)} \]

Example of lattice $U(1)$ gauge theory in 2+1d:

**Compact QED**

Consider: 2+1d system, notation $\Psi = \varphi + 2\pi$ defined on links

$[\Psi_p, n_p] = i \delta_{p,p'}$  $n_p =$ conjugate variable = integer

$0 \in [\varphi, 2\pi)$, $n$ = “angular momentum” if $\varphi =$ coordinate of particle on a ring

$e^{-i\hat{\varphi}} e^{i\hat{n}} = \hat{n} + m$
Gauss law: \( \nabla \cdot \mathbf{E} = 0 \)

Gauge invariant: \( [H, G_5] = 0 \)
\[
H = \alpha \sum_p n_p \quad \text{wouldn't have a bounded spectrum from below}
\]
\[
\rightarrow \frac{1}{2} kE \sum_p n_p^2 : \text{leading electric term}
\]

Since \( \Theta \) is angle, consider operator \( e^{i \theta_p} \rightarrow \) not gauge invariant
\[
e^{i \left( \Theta_{12} - \Theta_{23} + \Theta_{34} - \Theta_{14} \right)}
\]
\[
\rightarrow \text{gauge invariant object}
\]

To get rid of these signs, let's orient the lattice

\[
\begin{align*}
e_{ij} &= \varepsilon_i \varepsilon_j \\
a_{ij} &= \varepsilon_i \Theta_{ij}
\end{align*}
\]

Gauge constraint: \( \nabla \cdot \mathbf{E} = 0 \) (Gauss law)

\( \mathbb{Z}/2 \rightarrow \text{charges would be quantized} \)

For Compact QED

Gauge group: \( U(1) \)

Gauge invariant object: \( e^{i (\nabla \cdot a) \cdot \vec{n}} \) Magnetic flux

\[
L = a_{12} + a_{13} + a_{34} + a_{14}
\]
\[
H = \frac{K_E}{2} \sum_{\text{\(e\)}} e_p^2 - K_B \sum_{\text{\(e\)}} \cos \theta_0
\]

- \(K_E \gg K_B\): \(e_p \approx 0\) electric lines costly, confined phase satisfies constraint
- \(K_B \gg K_E\): favors \(\theta_0\) small (mod 2\(\pi\))
  \[\cos \theta_0 \approx 1 - \theta_0^2/2\]
  \[H \approx \frac{K_E}{2} \sum_{\text{\(e\)}} e_p^2 + \frac{K_B}{2} \sum_{\text{\(e\)}} \theta_0^2 + \ldots \sim \text{Usual QED} \text{ gapless photons}\]

Similar construction in any dimension.

**BUT:** in 2+1d, funneling between minima of \(\cos \theta_0\) crucial!
(Polyakov) \(\rightarrow\) confined phase only, \(\text{monopoles (Compact QED)}\)
\(\rightarrow\) photon gets a mass