The perturbative renormalization group

Idea: small parameter $\varepsilon \ll 1$ such that non-trivial fixed point is close to a "trivial" (say Gaussian) fixed point when $\varepsilon$ is small. $\Rightarrow$ controlled calculation

Single coupling example

$$\frac{du}{dp} = -Ku(u-u^*) + ...$$

$u$ is marginal
- for $\varepsilon = 0$, and relevant
- for $\varepsilon > 0$

say $K u^* > 0$: $u = 0$ unstable
$u^* = O(\varepsilon)$: perturbative stable
fixed point

Scaling operator at $u = 0$ fixed point with coupling $\lambda$:

$$\frac{d\lambda}{dp} = \gamma_\lambda \lambda + \gamma u \lambda + ...$$

$\gamma_\lambda \rightarrow \gamma_\lambda + \gamma u^* = O(\varepsilon)$ at new fixed point!

$\Rightarrow O(\varepsilon)$ corrections to critical exponents

(A) Operator product expansions and RG flow equations

(A.1) The operator expansion

Bring two scaling operators/fields close enough: "bare operators" $\Phi_i, \Phi_j \sim \sum_n c_n S_n$ around $\hat{\pi}_1, \hat{\pi}_2$

But each $S_n$ can be decomposed onto the scaling fields $\Phi_k$!
We thus expect: \[ \phi_i(\vec{n}_1) \phi_j(\vec{n}_2) = \sum_K c_{ijk} (\vec{n}_1 - \vec{n}_2) \phi_k(\vec{n}_1 + \vec{n}_2) \]

Valid when inserted in a correlation function \[ \langle ... \phi \rangle \] with \( p \gg |\vec{n}_1 - \vec{n}_2| \).

Physically: \( \phi_i, \phi_j \) viewed from far away looks like a local quantity, which may be expanded on the \( \phi_k \)'s.

Strictly speaking, we should also include derivatives \( \partial \phi_k \) but not important for our purposes.

Using \( \phi_i(\lambda \vec{n}) = \lambda^{-\Delta_i} \phi_i(\vec{n}) \):

\[ c_{ijk}(\vec{n}_1 - \vec{n}_2) = \frac{c_{ijk}}{|\vec{n}_1 - \vec{n}_2|^{\Delta_i + \Delta_j - \Delta_k}} \]

We will write:

\( \phi_i \times \phi_j \sim \sum_K c_{ijk} \phi_k \)

\[ [\phi_i] = L^{-\Delta_i} \]

A.2 The perturbative RG

Consider:

\[ S = S_0 + \sum_i g_i \int \frac{d^d \vec{n}}{a^{d-\Delta_i}} \phi_i(\vec{n}) \]

RG fixed point action/Hamiltonian

Here we introduced \( \sum_{\vec{n}} = \int \frac{d^d \vec{x}}{a^d} \) with \( a \) = lattice spacing/UV cutoff.
Perturbative expansion assuming $g_i \ll 1$ and $Z_0 = \int D\phi e^{-S_0}$, "easy"/known

$$Z = Z_0 \left[ 1 - \sum_i g_i \int {d^d \phi_i \over a_{Di} \Delta_i} \langle \phi_i(\vec{r}_i) \rangle + {1 \over 2!} \sum_{i,j} g_i g_j \int {d^d \phi_i \over a_{Di} \Delta_i} \int {d^d \phi_j \over a_{Dj} \Delta_j} \langle \phi_i(\vec{r}_i) \phi_j(\vec{r}_j) \rangle + \cdots \right] \quad \text{and} \quad \langle \cdots \rangle_0 = \int D\phi e^{-S_0} \langle \cdots \rangle_0 \over Z_0$$

$$\mathcal{O}(g^3)$$

. We already adopted a continuum picture: lattice effects unimportant, but remember the lattice for short-distance (UV) divergences of integrals. 

\[ \text{Cutoff regularization by requiring } |\vec{r}_i - \vec{r}_j| > a \] (Crude hard core cutoff to keep)

\[ \text{a continuum approach. End result won't depend on this} \]

. Even with this UV cutoff $a$, perturbation theory near a critical point is still singular (cf previous chapters) \[ \Rightarrow \text{introduce an infrared cutoff } \Lambda \]

. All integrals are then finite: system in a finite box $V = L^d$

$$N = V_{ad} \approx \# \text{ "sites"}$$

\[ L \) in this box: $\langle \phi_i(\vec{r}_i) \rangle \neq 0 \quad \text{and} \quad \langle \phi_i(\vec{r}_i) \phi_j(\vec{r}_j) \rangle \neq 0 \] \[ = {\delta_{ij} \over |\vec{r}_i - \vec{r}_j|^{2D} \Delta_i} \]

\[ \text{RG Scheme:} \quad \text{Change the UV cutoff } a \rightarrow \theta a \text{ with } \theta = e = 1 + \mathcal{O}(g^2) \]

\[ \text{Keep } L (\text{or } V) \text{ fixed so } N \text{ decreases} \]

\[ \text{Ask how the couplings } g_i \text{ should be changed to} \]

\[ \text{preserve } Z \text{ (to order } \mathcal{O}(g^2) \text{)} \]
Two contributions: (will add up in $\beta$ functions since $\delta p \ll 1$)

1. $\frac{1}{a-d_i}$ terms $\rightarrow \frac{1}{a-d_i} \left( 1 - (d - D_i) \delta p \right)$ after rescaling

2. Change in cutoff of $\mathbb{B}$ integrals: $\int \frac{a(1+\delta p)}{a-d_i-d_j} \left( \pi_i - \pi_j \right) = \int \frac{a(1+\delta p)}{a} \left( \pi_i - \pi_j \right)$

Second term: use OPEs then $\delta p \ll 1$: $-\frac{1}{2} \sum_{i'j'k'} C_{i'j'k'} \int \frac{d\pi}{a} \frac{d\pi'}{a-d_i-d_j} \left\langle \phi_k(\vec{R}) \right\rangle_0 \frac{\Delta_k}{\pi_+^\perp}$

OPE coefficients from $S_0^*$

integral over $\pi$: $\int \frac{a(1+\delta p)}{2\pi d/\Gamma(d/2)} \frac{d\pi}{a} \frac{d\pi'}{a-d_i-d_j} \Delta_k = a \frac{d-d_i-d_j+\Delta_k}{a}$

This gives a contribution: $-\frac{1}{2} \sum_{i'j'k'} C_{i'j'k'} S_d \delta p \int \frac{d\pi}{a} \frac{d\pi'}{a-d_i} \left\langle \phi_k(\vec{R}) \right\rangle_0$

=> can be compensated by changing $g_k \rightarrow g'_k = g_k - \frac{S_d}{2} \sum_{i'j'} C_{i'j'} \bar{g}_i \bar{g}_j \delta p$

=> Renormalized couplings: $g'_k = g_k + \delta p \left[ \frac{(d - \Delta_k)}{y_k} g_k - \frac{S_d}{2} \sum_{i'j'} C_{i'j'} \bar{g}_i \bar{g}_j \right]$

in differential form:

$$\frac{dg_k}{d\rho} = \frac{(d - \Delta_k)}{y_k} g_k - \frac{S_d}{2} \sum_{i'j'} C_{i'j'} \bar{g}_i \bar{g}_j + \ldots$$

not important: $g_i \rightarrow \frac{2}{c_i} g_i$
OPE coefficients control the UV singularities of the integrals and hence the β functions. Higher order terms: 3 or more points within distance a in the integrals.

B) The \( \phi^4 \) theory (Ising model) in \( d \geq 4 - \epsilon \) dimensions

Landau-Ginzburg action:

\[
S = \int d^n x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{t}{a^2} \phi^2 + \frac{\nu}{a^{d-2}} \phi^4 \right]
\]

ϕ normalized so that this coefficient is \( = 1/2 \)

Leading order RG equations:

\[
\begin{align*}
\phi &= L^{2-d} \\
\beta L &= \beta_0 L^{4-d} \quad \text{so that } S \text{ is invariant}
\end{align*}
\]

\[
\begin{align*}
\gamma \tau &= 2 \\
\gamma_v &= 4 - d \\
&\Leftrightarrow \text{at the fixed point } \nu = \tau = 0
\end{align*}
\]

B.1 Gaussian fixed point: \( \phi = 1 + \delta \phi \):

\[
\begin{align*}
\frac{d\phi}{d\rho} &= 2\phi + \cdots \\
\frac{d\nu}{d\rho} &= (4-d)\nu + \cdots
\end{align*}
\]

\( \nu = \tau = 0 \) obvious fixed point

\[
S^* = \int d^n x \frac{1}{2} (\nabla \phi)^2
\]

Gaussian fixed point: invariant under RG if \( d > 4 \):

\( \nu \) is irrelevant, and there is one relevant (thermally) perturbation \( \tau \)

(Recall \( \tau \sim (T - T_c) \) within Landau theory). We have \( \gamma_\tau = 2 \)

\( \Delta = 1/2 \) consistent with mean field (\( \Delta = 1/\gamma_\tau \))

We also have \( \Delta = d - 2 \) \( \Rightarrow \langle \phi(G_1) \phi(G_2) \rangle \sim \frac{1}{|G_1 - G_2|^{d-2}} \) \( \Rightarrow \gamma = 0 \)

(by dimensional analysis) on compute Gaussian integral
The fact that $u$ is irrelevant for $d=4$ and the values of these exponents suggest that perhaps we could set $u=0$ and that the Ising model for $d>4$ is controlled by a Gaussian fixed point plus a small perturbation $-\phi^2$

This is wrong: Add a field $-A \int d^n \phi \phi(n)$ to the action

$\Rightarrow \gamma = \frac{d}{2} + 1$ so that

\[ \beta = \frac{d-2}{\eta} \]
\[ \alpha = 2 - d/2 \]
\[ \eta = 0 \]
\[ \delta = \frac{d+2}{d-2} \]

Exponents of Gaussian fixed point

$[\phi] = L^{2-d/2}$

Vs Mean Field

$\beta = \frac{1}{2}$
\[ \alpha = 0 \]
\[ \eta = 0 \]
\[ \delta = 3 \]

This agrees with MF only for $d=4$! This is because $u$ is a dangerously irrelevant variable: we can't set it to 0!

E.g.: ordered phase $\langle \phi \rangle \sim (-\frac{t}{u})^{1/2}$ by minimizing $S$: $u\neq 0$ needed!

$1/2 \times 0 < t < 0$

- Gaussian theory controls fluctuations over MF results, but exponents are less singular for $d>4$: MF contribution dominates.

$\int_{d<4} u$ is relevant

$\frac{du}{dp} = \varepsilon u + \ldots$ with $\varepsilon = 4-d$

$\Rightarrow$ new fixed point (Wilson-Fisher): "close" to Gaussian fixed point if $\varepsilon \ll 1$ ⇒ controlled calculation in $d=4-\varepsilon$ dimensions!

(in practice: set $\varepsilon = 1$ at the end of the calculation)
B.2 Wick Theorem and OPEs of the Gaussian Theory

All we need is the OPE structure of

\[ S_0^* = \frac{1}{2} \int d^n \phi \left( \nabla \phi \right)^2 \]

Because \[ [\phi] = L^{2-d} \]

\[ \langle \phi(p_1) \phi(p_2) \rangle_0 = \frac{1}{|p_1 - p_2|^{d-2}} \]

set to vanish by redefining the normalization of \( \phi \)

\[ S_0^* : \text{Higher order correlation functions given by Wick's Theorem} \]

\[ \langle \phi(p_1) \phi(p_2) \phi(p_3) \phi(p_4) \rangle_0 = \langle \phi(p_1) \phi(p_2) \rangle_0 \langle \phi(p_3) \phi(p_4) \rangle_0 + \langle \phi(p_1) \phi(p_3) \rangle_0 \langle \phi(p_2) \phi(p_4) \rangle_0 \]

\[ + \langle \phi(p_1) \phi(p_3) \rangle_0 \langle \phi(p_2) \phi(p_4) \rangle_0 \]

Also holds if some points coincide:

\[ \langle \phi^2(p_1) \phi^2(p_2) \rangle_0 = 2 \langle \phi(p_1) \phi(p_2) \rangle_0^2 + \langle \phi^2 \rangle_0^2 \]

\[ \text{independent of } |p_1 - p_2| \]

\[ \Rightarrow \text{it is convenient to use a slightly different basis:} \]

\[ \phi^2 = \phi^2 - \langle \phi^2 \rangle_0 \]

\[ \phi^4 = \phi^4 - 3 \langle \phi^2 \rangle_0 \phi^2 \]

\[ \ldots \]

\[ \text{Normal ordered } \]

\[ \text{Scaling operators } \Rightarrow \text{no Wick contractions at the same point} \]

\[ \text{work with these operators in } S \text{ from the beginning} \]

\[ \text{OPEs: } \phi^2(p_1) : \phi^2(p_2) : = \frac{2}{|p_1 - p_2|^{2-d}} + \frac{4}{|p_1 - p_2|^{d-2}} : \phi^2(p_1 + p_2) : + \phi^4(p_1 + p_2) : + \ldots \]

\[ \text{We write: } \phi^2 : x : \phi^2 : \sim 2 + 4 : \phi^2 : + \phi^4 : \]
Other OPE's follow from simple combinatorics:

\[
\phi^2 \times \phi^4 \sim 4 \times 3 \phi^2 + 8 \phi^4
\]

\[
\phi^4 \times \phi^4 \sim 4 \times 3 \times 2 \left(\frac{4 \times 3 \times 2}{3!}\phi^2 + \frac{(3 \times 4)^2}{2!}\phi^4\right)
\]

Choose 3 on each side, choose \(\frac{(4 \times 3)^2}{2}\) possibilities, overcounts by 3! (permutations of the -lines)

We thus have:

\[
\phi^2 \times \phi^2 \sim 2 + 4 \phi^2 + \phi^4
\]

\[
\phi^2 \times \phi^4 \sim 12 \phi^2 + 8 \phi^4
\]

\[
\phi^4 \times \phi^4 \sim 24 + 96 \phi^2 + 72 \phi^4
\]

\[8\]

### B.3 Wilson-Fisher Fixed Point

We can immediately write down the RG flow equations:

\[
\frac{dt}{dp} = 2t - 4t^2 - 2 \times 12 t + 96 \epsilon u^2 + \ldots
\]

\[
\frac{du}{dp} = \epsilon u - t^2 - 2 \times 8 t + 72 \epsilon u^2 + \ldots
\]

\[
C_{ijkl} = C_{jikl} \quad i \neq j
\]

New fixed point with:

\[
\epsilon = 0 + O(\epsilon^2)
\]

\[
u* = \frac{\epsilon}{72} + O(\epsilon^2)
\]

Thermal eigenvalue:

\[
\frac{dt}{dp} = 2t - 29u* + \ldots \Rightarrow \gamma_T = 2 - \frac{29}{72} \epsilon + O(\epsilon^2)
\]
Using $\nu = \frac{1}{4}$, we have

$$\nu = \frac{1}{2} + \frac{e}{12} + O(e^2)$$

Compute $\eta$ by introducing a field: $\phi \times \phi = 1 + \phi^2$:

$$\phi \times \phi^2 = 2\phi + \phi^3$$

$$\phi \times \phi^3 = 4\phi^3 + ...$$

$$\frac{d\eta}{dp} = (\frac{d}{2} + 1)\nu - 2 \times 2 \eta + ...$$

No $\eta$ term in $\frac{d\eta}{dp}$: $\eta = 0 + O(e^2)$ (\eta \neq 0 for higher orders)

\[\text{How useful is the } e \text{ expansion?} \]

Set $e = 1$ and $e = 2$ for $d = 3$ and $d = 2$

\begin{align*}
\text{d=2: } & \eta = 0, \quad \nu_{O(e^1)} = 0.66 \\
\text{d=3: } & \eta_{O(e^1)} = 0.58, \quad \nu_{O(e^1)} = 0.58
\end{align*}

For $d = 3$, not too bad. Can be improved by going to $O(e^3)$

\begin{align*}
\text{d=2: } & \eta \approx 0.26, \quad \nu_{O(e^3)} \approx 0.94 \\
\text{d=3: } & \eta \approx 0.037, \quad \nu_{O(e^3)} \approx 0.63
\end{align*}

Requires extrapolations: $e$-series asymptotic only
Remark: in retrospect, we only need two OPE coefficients:

\[ \phi^2 \times \phi^4 = 12 \phi^3 + \ldots \]

\[ \phi^4 \times \phi^4 = 72 \phi^5 + \ldots \]

Feynman diagrams (not used here)

\[ \phi^2 \]

(not quite right)

\[ \phi^4 \]

\[ \phi^4 \times \]

\[ \rightarrow \]

B.4 Irrelevant operators: we can now justify why we could ignore terms like \( \int d^n g_{2n} \phi^{2n} \) in the action.

\[ \begin{align*}
[\phi] &= \mathcal{L} \frac{\partial}{\partial \phi} \\
[\phi^{2n}] &= \mathcal{L} \Delta_{\phi^{2n}} = n(d-2)
\end{align*} \]

\[ \Rightarrow \]

\[ \frac{dg_{2n}}{dp} = (d - n(d-2)) g_{2n} + \ldots \]

Near Gaussian fixed points

If \( d \) is near 4, \( \Delta_{\phi^{2n}} > 4 \) for \( n \geq 3 \)

\[ \Rightarrow \]

\[ \text{irrelevance.} \]

For \( d \leq 4 \), we should check that \( \phi^6, \phi^8 \ldots \) are irrelevant at the WFT fixed point:

\[ \phi^4 \times \phi^{2n} = 4 \times 3 \times \frac{2n(2n-1)}{2!} \phi^{2n} + \ldots \]

\[ \Rightarrow \]

\[ \frac{dg_{2n}}{dp} = (d - n(d-2)) g_{2n} - 2 \times 12 n (2n-1) u g_{2n} + \ldots \]

So that

\[ \gamma_{2n} = d - n(d-2) - 24 n (2n-1) u^* + o(u^2) \]

\[ = 4 - 2n + o(n-1) - \frac{n(2n-1)}{3} \epsilon + o(\epsilon^2) \]

\[ \text{connection due to } u^* \text{: even more irrelevant at Wilson Fisher fixed point} \]

\[ \Rightarrow \text{contrary to real space approach, this field theory ("continuum") calculation starting from Landau-Ginzburg is bully controlled!} \]
The $O(n)$ model in $d = 4 - \varepsilon$ dimensions

So far, we have focused on the universality class of the Ising model. To illustrate the generality of the approach, we now consider the $O(n)$ model

$$ S = \int d^n x \left[ \frac{1}{2} (\nabla \Phi)^2 + \frac{t}{a^2} \Phi^2 + \frac{u}{a^{d-1}} (\Phi^2)^2 + \ldots \right] $$

with $\Phi$: $n$ component vector
($n=1$: Ising)

C.1 $\varepsilon$ expansion

Dimensional analysis is the same as Ising, and $\langle \phi_i, \phi_j \rangle_0 = \frac{\delta_{ij}}{\nu d-2}$ at the Gaussian fixed point ($t=0, u=0$).

We need two OPE coefficients:

$\Phi^2 \times \Phi^2 : \Phi^2 + \ldots$

$\Phi^4 \times \Phi^4 : \Phi^4 + \ldots$

$\sum_{i,j,k} \delta_{ij} \delta_{ik} \text{ index on n: } 2 \times 2 \times 2$

$\sum_{i,j,k} \delta_{ij} \delta_{ik} \text{ index on n: } 2 \times 2 \times 2$

$\sum_{i,j,k} \delta_{ij} \delta_{ik} \text{ index on n: } 2 \times 2 \times 2$

This yields:

$$ \frac{du}{dp} = \varepsilon u - 8(n+8) u^2 + \ldots $$

$$ \frac{dt}{dp} = 2t - 8(n+2) u t + \ldots $$

Fixed point at $u^* = \frac{\varepsilon}{8(n+8)}$

$\gamma = 2 - \frac{n+2}{n+8} \varepsilon + O(\varepsilon^2)$

$\gamma = \frac{1}{2} + \frac{1}{4} \frac{n+2}{n+8} \varepsilon + O(\varepsilon^2)$
Large $n$ limit of the $O(n)$ model

For $n \to \infty$, $u^* \to 0$ suggests $\gamma_f = 2 - \varepsilon = d - 2$ exact in that limit.

Can be shown explicitly: change our notation slightly:

$$ S = \frac{1}{2} \int dx \left[ (\nabla \phi)^2 + t \phi^2 + \frac{u}{n} (\phi^2)^2 \right] $$

Now use trick: $e^{-\frac{1}{2} \int dx \frac{u}{n} (\phi^2)^2} = \int D\phi e^{-\frac{1}{2} \int dx \left( \sigma^2 + i \sqrt{\frac{u}{n}} \sigma \phi^2 \right)}$

Hubbard-Stratonovich transformation (up to unimportant constants)

Can be shown by completing the square:

$$(\sigma + i \sqrt{\frac{u}{n}} \phi^2)^2 + \frac{u}{n} \phi^2$$

$Z = \int D\sigma D\phi e^{-\frac{1}{2} \int dx \sigma^2 - \frac{1}{2} \int dx \sum_i \phi_i \left( -\nabla^2 + t + i \sqrt{\frac{u}{n}} \sigma(x) \right) \phi_i}$

we can integrate out $\phi$: Gaussian integral!

$$ = \int D\sigma e^{-\frac{1}{2} \int dx \sigma^2} \left( \text{Det} \left[ -\nabla^2 + t + i \sqrt{\frac{u}{n}} \sigma \right] \right)^{-\frac{1}{2}}$$

Note: we're thinking (formally) of $-\nabla^2 + t + i \sqrt{\frac{u}{n}} \sigma$ as an operator $M$ that appears in the Gaussian integral: $\int dx dx' \phi(x) M(x', x) \phi(x)$, "Matrix product" with $M(x', x) = \delta(x' - x) \left( -\nabla^2 + t + i \sqrt{\frac{u}{n}} \sigma(x) \right)$

Eigenvalue $\langle \tilde{x}' | (-\nabla^2 + t + i \sqrt{\frac{u}{n}} \sigma) | \tilde{x} \rangle$ using QM notations

$$\text{Det} M = \prod_k \lambda_k = \exp \left( \log \prod_k \lambda_k \right) = \exp \left( \sum_k \log \lambda_k \right) = \exp \left( T \sum \log \lambda_k \right)$$
Rescaling $\sigma \to \sqrt{u} \sigma$, we get:

$$Z = \int D\sigma \, e^{-\beta S_{\text{eff}}[\sigma]}$$

with

$$S_{\text{eff}}[\sigma] = \frac{1}{2} \int d^d x \sigma^2(x) + \frac{1}{2} \, \text{Tr} \log \left( \frac{(-\nabla^2 + t + i\sqrt{4u} \sigma)^{-1}}{M^{-1}} \right) \int d^d x \, M^{-1}(x,x) \, i\sqrt{4u} \sigma(x)$$

Effective action for auxiliary $\sigma$ field

(\text{path integral}) integral dominated by saddle point

**Saddle point**: $S_{\text{eff}} = \int d^d x \, \sigma(x) \, \delta\sigma(x) + \frac{1}{2} \, \text{Tr} \left[ \frac{(-\nabla^2 + t + i\sqrt{4u} \sigma)^{-1}}{M^{-1}} \right] \int d^d x \, M^{-1}(x,x) \, i\sqrt{4u} \sigma(x)$

Here $M^{-1}$ is the Green's function of $(-\nabla^2 + t + i\sqrt{4u} \sigma)$:

$$\frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = 0 = \sigma_{\text{eff}} + i\sqrt{u} \left( \int \frac{d^d K}{(2\pi)^d} \frac{1}{K^2 + t + i\sqrt{4u} \sigma} \right)$$

where we assumed $\sigma = \text{cst}$

(look for uniform solution)

Let $\tau_{\text{eff}} = t + i\sqrt{4u} \sigma$ (assuming $\sigma \in i\mathbb{R}$)

$$\Rightarrow \tau_{\text{eff}} - t = 2u \int \frac{d^d K}{(2\pi)^d} \frac{1}{K^2 + \tau_{\text{eff}}}$$

UV cutoff: underlying lattice

**UV cutoff**: underlying lattice

Equation for $\tau_{\text{eff}}$

$$1/K^2 = \frac{\tau_{\text{eff}}}{K^2(K^2 + \tau_{\text{eff}})}$$

Repeating the mapping for the expectation value $\langle \Phi^2 \rangle$, one can show that there is a phase transition at $\tau_{\text{eff}} = 0$ (and $\tau_{\text{eff}} \sim \Phi^{-2}$): if $\tau_{\text{eff}} < 0$, $\langle \Phi^2 \rangle \neq 0$

Critical temperature $t = t_c = -2u \int \frac{d^d K}{(2\pi)^d} \frac{1}{K^2}$

Critical $T$ shifted down (vs Mean field: $t_c = 0$)

$$\Rightarrow \tau_{\text{eff}} = t - t_c - 2u \tau_{\text{eff}} \int \frac{d^d K}{(2\pi)^d} \frac{1}{K^2(K^2 + \tau_{\text{eff}})}$$

$1 \rightarrow$ convergent if $d > 2$

(If $d < 2$: no transition: Mean Field!)
Now:
\[ \int \frac{\Delta K}{(2\pi)^d} \frac{1}{K^2(K^2 + \Lambda^2)} \propto \int \frac{1}{\sqrt{\Lambda^2 K}} dK \propto \frac{1}{K} \]

\[ \Rightarrow K \to \sqrt{\Lambda^2 K} \]

\[ \int \frac{dK}{K} - 2 \]

\[ = \begin{cases} 
\Lambda^{d-2} \times \text{cst} & \text{if } d < 4 \Rightarrow \text{integral UV convergent we can send } \Lambda \to \infty \\
\Lambda^{d-4} \sim \Lambda^{d-2} & \text{if } d > 4 \Rightarrow \text{integral UV divergent scales as } \Lambda^{d-2} \text{ independent of } \Lambda^2 
\end{cases} \]

\[ \text{if } d > 4: \ t_{\text{eff}} \sim t - t_c \text{ and } t_{\text{eff}} \sim \frac{1}{\Lambda^{d-2}} \Rightarrow \gamma = \frac{1}{2} \text{ MF result } \]

\[ \text{if } d < 4: \ t_{\text{eff}} \sim (t - t_c)^{\frac{2}{d-2}} \sim \frac{1}{\Lambda^{d-2}} \Rightarrow \gamma = \frac{1}{d-2} \text{ consistent with } \epsilon \text{ expansion, exact for } n \to 0 \]

\[ t_{\text{eff}} = t - t_c - \text{cst} \ t_{\text{eff}}^{d-1} \]

\[ \text{negligible as } t_{\text{eff}} \to 0 \]

Can be used as a starting point for the so-called large \( n \) expansion

\[ \Rightarrow \text{compute corrections in } \frac{1}{n} \text{ and then set } n = 1, 2, 3 \text{ (usual physical values)} \]
Appendix: Dangerously irrelevant variable in $\phi^4$ theory for $d > 4$

For $d > 4$, we expect fluctuations to be small and MF exponents to be correct ($\kappa = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3, \eta = 0, \nu = \frac{1}{2}$).

The coupling $\nu \phi^4$ is irrelevant but $d > 4 \Rightarrow$ critical behavior controlled by Gaussian fixed point? (set $\nu = 0$) But some of the Gaussian theory exponents are different!

- e.g. $\beta = \frac{d-2}{q}$ at Gaussian fixed point ($\gamma_+ = 2, \gamma_- = 1 + \frac{d}{2}$)

$\Rightarrow$ this is because $\nu \phi^4$ is dangerously irrelevant.

**RG equation for $m$:**

$$m(t, R, \nu) = e^{n(y_R - d)} F(t^{(y_R - d)/\gamma_+}, R^{\gamma_+ / \gamma_+}, \nu e^{n\gamma_+})$$

Set $R = 0, e^{n\gamma_+ t} = O(1) \Rightarrow m(t, 0, \nu) = t^{-(y_R - d)/\gamma_+} \Phi(u + \nu/\gamma_+)$$

Naively: $t^{-\gamma_+/\gamma_+} \to 0$ if $d > 4$, and $\Phi(0)$ finite $\Rightarrow m \sim t^{-(y_R - d)/\gamma_+}$

(effectively: set $\nu = 0$)

But: Landau theory predicts $m \propto \sqrt{t} \Rightarrow$ can't set $u = 0$

$$m(t, 0, \nu) \propto u^{-1/2} \Rightarrow \Phi(x) \sim x^{-\nu/\gamma_+} \sqrt{x}$$

Limit $x \to 0$ ($u \to 0$) not smooth!

Then

$$m(t, 0, \nu) = \left| t \right|^{-(y_R - d)/\gamma_+} \Phi\left( \left| t \right|^{-\gamma_+ / \gamma_+} \nu / \gamma_+ \right) \sim \left| t \right|^\beta$$

With

$$\beta = \frac{d - y_R}{\gamma_+} + \frac{\gamma_+}{2\gamma_+} = \frac{d-2}{q} + \frac{4-d}{q} = \frac{1}{2} \Rightarrow \beta = \frac{1}{2}$$

$\Rightarrow$ we recover MF result as expected.