Smarr Formula and Extended First Law for AdS Black Holes

Includes variations in Λ

Thermodynamic parameter conjugate to Λ

$$dM = \frac{\kappa}{8\pi} dA + \frac{\Theta}{8\pi} d\Lambda$$

Kastor, Ray & Traschen, "Enthalpy and the Mechanics of AdS Black Holes," arXiv:0904.2765

Kastor, Ray & Traschen, "Smarr Formula and an Extended First Law for Lovelock Gravity," arXiv:1005.5053

<u>Outline</u>

1) Euler equation in classical thermodynamics

2) Obtaining Smarr formula with $\Lambda = 0$

3) $\Lambda \neq 0$ via "enhanced" Komar integral relations

Killing potential is new ingredient

1) Euler equation in classical thermodynamics

Smarr formula
is GR analogue

$$dE = TdS - PdV$$ 1st Law

can express

$$E = E(S,V) \quad \text{with} \quad T = \frac{\partial E}{\partial S}, \qquad P = -\frac{\partial E}{\partial V}$$

E, S and V are all extensive variables

applying Euler's theorem for homogeneous functions....

$$f(\alpha^p x, \alpha^q y) = \alpha^r f(x,y) \quad \Rightarrow \quad r f(x,y) = p\left(\frac{\partial f}{\partial x}\right) x + q\left(\frac{\partial f}{\partial y}\right) y$$

Gives Euler equation

$$\boxed{E = TS - PV}$$

Combine with 1st law to obtain
Gibbs-Duhem formula

$$0 = SdT - VdP$$

Relation between
intensive variables

2) Smarr formula with $\Lambda = 0$

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ$$

1st law for stationary black holes

Redo scaling argument in D dimensions …

$$M \propto L^{D-3}, \qquad A \propto L^{D-2}, \qquad J \propto L^{D-2}$$

Applying Euler's theorem to $M(A, J)$ gives Smarr formula

$$rf(x,y) = p\left(\frac{\partial f}{\partial x}\right) x + q \left(\frac{\partial f}{\partial y}\right) y$$

$$(D-3)M = (D-2)\frac{\kappa A}{8\pi} + (D-2)\Omega J$$

2) Smarr formula with $\Lambda = 0$

Can also derive Smarr formula via Komar integral relation

ξ^a Killing vector

Let $$B^{ab} = \left(\frac{D-2}{8\pi}\right)\nabla^a\xi^b$$

$$\nabla_a\nabla^a\xi^b = -R^b{}_c\xi^c$$

then $$B^{ab} = B^{[ab]}$$

$$\nabla_a B^{ab} = 0$$ For vacuum spacetimes

On a spatial slice extending from the horizon to infinity $$I = \int_{\partial\Sigma} dS_{ab}B^{ab} = 0$$

For a static black hole $$I_\infty = (D-3)M$$

$$I_H = (D-2)\frac{\kappa A}{8\pi}$$

→ **Smarr formula**

3) Smarr formula with $\Lambda \neq 0$

Focus on static black holes

Apply Euler's theorem to $M = M(A, \Lambda)$

scalings $M \propto L^{D-3}, \qquad A \propto L^{D-2}, \qquad \Lambda \propto L^{-2}$

defining $$\frac{\Theta}{8\pi} \equiv \frac{\partial M}{\partial \Lambda}$$

gives $$(D-3)M = (D-2)\frac{\kappa A}{8\pi} - 2\frac{\Theta\Lambda}{8\pi}$$

Form of Smarr formula

$$dM = \frac{\kappa}{8\pi}dA + \frac{\Theta}{8\pi}d\Lambda$$

Extended 1st law

New term accounts for change in mass when Λ is varied.

What is Θ?

3) Smarr formula with $\Lambda \neq 0$

$$(D-3)M = (D-2)\frac{\kappa A}{8\pi} - 2\frac{\Theta\Lambda}{8\pi}$$

$$dM = \frac{\kappa}{8\pi}dA + \frac{\Theta}{8\pi}d\Lambda$$

What is Θ?

Can compute explicitly for Schwarzschild-AdS spacetimes ← Reasonable first step, but doesn't reveal geometric significance.

Caldarelli et. al. (1999)
Wang et. al. (2006)
Sekiwa (2006)
Wang (2006)
Larranaga Rubio (2007)
G. Cardoso & Grass (2008)
Urano et. al. (2009)

We give geometric derivations of….

Extended 1st law using Hamiltonian perturbation theory techniques — Sudarsky & Wald (1992)

Smarr formula via enhanced Komar integral relation — Easier

3) Smarr formula for AdS black holes via enhanced Komar integral relation

Killing potential

Introduce Killing potential $\nabla_a \xi^a = 0 \Rightarrow \xi^b = \nabla_a \omega^{ab}$

Einstein equation $R_{ab} = \frac{2\Lambda}{(D-2)} g_{ab} \Rightarrow \nabla_a \nabla^a \xi^b = -\frac{2\Lambda}{D-2}\xi^b$

New Komar integral relation

$$I = \frac{D-2}{8\pi}\int_{\partial\Sigma}\left(\nabla^a \xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right) = 0$$

$$0 = \frac{D-2}{8\pi}\left[\int_\infty dS_{ab}\left(\nabla^a \xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right) - \int_H dS_{ab}\left(\nabla^a \xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right)\right]$$

Both these terms diverge at infinity due to AdS asymptotics, but sum is finite.

Need to tease apart contributions to M and ϑ.

$$0 = \frac{D-2}{8\pi}\left[\int_\infty dS_{ab}\left(\nabla^a\xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right) - \int_H dS_{ab}\left(\nabla^a\xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right)\right]$$

For pure AdS there is no horizon integral.
The two divergent terms cancel precisely.

Suggests adding and subtracting ω^{ab}_{AdS} ← Pure AdS Killing potential

Gives $(D-3)M$ — Contributes to Θ

$$0 = \frac{D-2}{8\pi}\left[\int_\infty dS_{ab}\left(\left[\nabla^a\xi^b + \frac{2\Lambda}{D-2}\omega^{ab}_{AdS}\right] + \frac{2\Lambda}{D-2}\left[\omega^{ab} - \omega^{ab}_{AdS}\right]\right) - \int_H dS_{ab}\left(\nabla^a\xi^b + \frac{2\Lambda}{D-2}\omega^{ab}\right)\right]$$

Gives $(D-2)\frac{\kappa A}{8\pi}$

Contributes to Θ

Note that we've done a background subtraction and a background addition obtain $$(D-3)M = (D-2)\frac{\kappa A}{8\pi} - 2\frac{\Theta\Lambda}{8\pi}$$

with $$\Theta = \int_H dS_{ab}\omega^{ab} - \int_\infty dS_{ab}(\omega^{ab} - \omega^{ab}_{AdS})$$

Derivation of extended 1st law yields same result

$$dM = \frac{\kappa}{8\pi}dA + \frac{\Theta}{8\pi}d\Lambda$$

What is Θ? → "renormalized" difference of surface integrals of the Killing potential between horizon and infinity.

A few remarks …

$$(D-3)M = (D-2)\frac{\kappa A}{8\pi} - 2\frac{\Theta\Lambda}{8\pi}$$

$$dM = \frac{\kappa}{8\pi}dA + \frac{\Theta}{8\pi}d\Lambda$$

$$\Theta = \int_H dS_{ab}\omega^{ab} - \int_\infty dS_{ab}(\omega^{ab} - \omega^{ab}_{AdS})$$

1. ϑ is independent of ambiguities in choice of Killing potential

$$\xi^b = \nabla_a \omega^{ab}$$

2. For AdS-Schwarzschild one finds….

$$\Theta = -\frac{\Omega_{D-2} r_H^{D-1}}{D-1} = -V_H$$

Volume inside a sphere of radius r_H in AdS

Agrees with previous calculations of

$$\Theta \equiv 8\pi \frac{\partial M}{\partial \Lambda}$$

A few remarks …

$$(D-3)M = (D-2)\frac{\kappa A}{8\pi} - 2\frac{\Theta\Lambda}{8\pi}$$

$$dM = \frac{\kappa}{8\pi}dA + \frac{\Theta}{8\pi}d\Lambda$$

$$\Theta = \int_H dS_{ab}\omega^{ab} - \int_\infty dS_{ab}(\omega^{ab} - \omega^{ab}_{AdS})$$

3. New term in first law relevant in a scenario with varying Λ.

• e.g. scenario of Brown & Teitelboim (1987) for neutralizing Λ through nucleation of pairs of branes charged under a D-form field strength.

• Perhaps there is some application for a thermodynamic ensemble at fixed ϑ, rather than fixed Λ?

4. Can similarly derive Smarr formula and extended first law for asymptotically AdS black holes in Lovelock gravity.

Includes variations in all the Lovelock coupling constants.

• Perhaps interesting in the context of gauge-gravity duality with Gauss-Bonnet or Lovelock gravity in the bulk.

Bulk Lovelock couplings related to conformal anomaly terms on boundary.