Phylogeny and Ontogeny of Mathematical and Numerical Understanding

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Abstract and Keywords

This navigator chapter situates the chapters that comprise the section on the phylogeny and ontogeny of mathematical and numerical understanding. How is number represented in the absence of language? What are the key questions that arise as we map out the continuities and discontinuities between non-human and human numerical cognition? What can we learn from studying individual differences in numerical cognition? How do the initial representations of quantity in the infant give rise to the uniquely human mathematical mind? Can we use the knowledge we are gaining about how the preverbal mind represents and manipulate quantity to improve mathematics education?

Keywords: approximate number system, evolution, development, ratio-dependence, object-file system, Weber's Law

Introduction

The case study of numerical cognition has proven to be a remarkable test bed for understanding the evolution and development of the mind. Comparative psychologists seek to understand what differentiates human minds from animal minds, and importantly the study of how the mind represents number illustrates both evolutionary continuity and discontinuity. On the one hand, the continuity is evident in a shared system for making approximate numerical judgments (i.e. the approximate number system or ANS). On the other hand, there is a clear evolutionary discontinuity in that only the human species has invented arbitrary symbols for number. This unique capacity for representing number symbolically allows humans to mentally manipulate precise numerical values and permits complex and abstract mathematics. In parallel, developmental psychologists seek to understand the origins of human conceptual abilities. The study of how number is represented from infancy into adulthood reveals profound continuity whereby infants appear to enter the world able to represent and compare numerical values approximately. While this system for representing number approximately dramatically improves in its precision over early childhood and into middle age, its fundamental signatures remain constant. At the same time, numerical development reveals a paradigmatic example of discontinuity and conceptual change whereby language transforms a child's ability to represent number (Carey 2009). The chapters in this section represent new and exciting research on both the development and the evolution of numerical thinking and highlight cutting edge questions in cognitive science that emerge from this field.

Evolutionary and developmental continuity

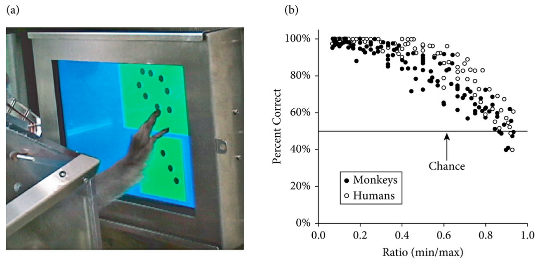

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Figure 1 Monkeys show ratio dependence when ordering numerosities. (a): A photograph of a monkey engaging in a numerosity ordering touch-screen task. (b): Accuracy as a function of the ratio between the two numerical values. Open symbols are data from college students and closed symbols are data from rhesus monkeys. Ratio refers to small/large numerosity. Reprinted from Cantlon and Brannon 2006.

Species throughout the animal kingdom are sensitive to the numerical attributes of the world around them (Agrillo, this volume; Beran, this volume). The primary characteristic of animal number representations is that they are imprecise. Discrimination follows Weber's Law whereby the ability to discriminate between two values is determined by their ratio not their absolute difference. Figure 1 illustrates that accuracy in a numerical comparison task is predicted by the ratio of the two numerical values being compared. In this task the animal is presented with two numerosities on a touch sensitive screen and is required to choose the array with the larger number of dots. In fact, Figure 1 shows that when adult humans are tested in the same task that prohibits verbal counting, their performance is indistinguishable from that of nonhuman animals (see also Agrillo et al. 2012; Beran et al. 2011a; Beran et al. 2008, 2011b; Cantlon & Brannon, 2007; Jordan & Brannon 2006a,b). This pattern of results suggests that we share with many diverse species a system for representing number approximately, which is referred to as the approximate number system (ANS).

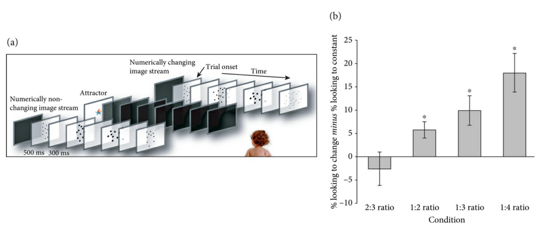

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Figure 2 Infant numerical change detection tasks demonstrates ratio dependence. (a): Experimental design of numerical change detection task. During each trial, infants were presented with two image streams simultaneously on two peripheral screens. In the numerically changing image stream, images contained two different numerosities in alternation (here: 10 and 20), while the numerically non-changing image stream contained only images with the same numerosity (here: 10). Infants' looking time to each of the image streams was measured. Reprinted from Libertus & Brannon 2010. (b): six-month-old infants' preference scores for numerically changing image streams in four different ratio conditions. Significantly positive preference scores were found for the 1:2, 1:3, and 1:4 ratio conditions. Furthermore, preference scores increased with increasing relative numerical disparity. Error bars reflect standard errors. Reprinted from Libertus & Brannon 2010.

Sensitivity to number is part of the human repertoire even at birth (McCrink, this volume; Izard et al. 2009). A recent set of studies shows parametric ratio dependence in the time infants spend looking at visual arrays as predicted by Weber's Law. As shown in Figure 2, infants were shown two simultaneous streams of visual arrays. One stream maintained a constant numerosity and the other stream alternated between two numerosities. Infants' preference for the changing stream depended on the ratio between the two values in the changing stream (Libertus & Brannon 2010; Starr et al. 2013a; see also Xu & Spelke 2000).

Particularly compelling evidence for the existence of this primitive number system in human infants and animals comes from studies that requires cross-modal matching of arrays based on number. Newborn infants preferentially look at a visual array that numerically matches the number of phonemes to which they have been familiarized (Izard et al. 2009). Rhesus monkeys and human infants presented with arrays of two and three conspecifics vocalizing allocate more visual attention to the array that numerically matches the number of conspecific calls or voices they hear (Jordan et al., 2005; Jordan & Brannon 2006c). Unfortunately, those studies did not address ratio-dependence and instead were limited to the small values 2 and 3. However, rhesus monkeys were tested with an operant task and a wide range of numerical values and were able to reliably choose a visual array that numerically matched an auditory sample sequence (Jordan et al. 2008). These cross-modal findings are one of the strongest arguments against the alternative explanation that humans and non-human animals are sensitive, not to number, but to perceptual variables such as surface area, density, or duration that are typically confounded with number.

Although the characteristic signatures of the ANS appear to be present very early in human development, the system undergoes a great deal of change over the lifespan. While newborns require a 1:3 ratio to detect a change in numerosity (Izard et al. 2009), by six months of age infants are capable of differentiating a 1:2 ratio change, and by nine months they succeed with a 2:3 ratio change (Xu & Spelke 2000; Lipton & Spelke 2004). Numerical comparison tasks indicate that precision in numersoity discrimination continues to improve from three years to thirty years of age (Halberda & Feigenson 2008; Halberda et al. 2012). Indeed, numerical cognition may follow the characteristic cognitive aging decline from 30 years onwards (Uittenhove & Lemaire, this volume).

ANS representations are ubiquitous across species and developmentally conservative, thus it seems likely that they have an important function throughout evolution. A candidate function for ANS representations that might account for such universality is their importance for arithmetic calculations (McCrink & Birdsall, this volume; Gilmore, this volume). Analog representations of number are far more useful than digital representations if we assume that organisms must integrate number, space, and duration to calculate rates and construct probabilities (Dehaene & Brannon 2011; Gallistel 2011). As reviewed by McCrink & Birdsall (this volume), infants show behavioral and ERP indications that they detect anomalous outcomes after observing arithmetic operations. Other studies indicate that preschoolers, older children, adults, and even monkeys appreciate ordinal relations between numerical values and perform approximate arithmetic on ANS representations (Gilmore, this volume).

As Agrillo describes (this volume), the ANS may not be the only way number can be represented without language. Infants and adults appear to use an alternative route for keeping track of small numbers of individual items, referred to as the object-file system (Uller et al. 1999; See Feigenson et al. 2004 for review). In adults this presents as exceedingly fast and accurate enumeration or subitizing of one to four items in contrast to the serial increase in reaction-time that is observed with each additional item to-be-enumerated for arrays of five or greater. Evidence for an object-file system in infancy comes from a few different experimental paradigms. In each of these nonverbal procedures infants seem only capable of tracking small numbers of objects and display a remarkable inability to compare small with large sets. Nevertheless, it must be emphasized that while these findings are robust and replicable, they are context specific. In other experimental contexts infants appear to be capable of representing small and large sets with a single cognitive system and even show ratio dependence when discriminating small sets (e.g. Starr et al. 2013a; vanMarle & Wynn 2011).

Agrillo describes research with a wide range of species from elephants, dogs, parrots, beetles, and primates. In some cases animals exhibit ratio dependence throughout the entire numerical range and in other cases they show the pattern described above for infants whereby they are either unable to represent large values or exhibit differences in the ratio required for small and large number discrimination. These data raise interesting questions about whether species differences or experimental parameters explain whether behavioral patterns are controlled by an object-file system or the ANS.

Conceptual foundations for uniquely human mathematical cognition

While the above section illustrates significant continuities in numerical cognition throughout evolution and development there are also important discontinuities (Carey 2009; Spelke 2011; Sarnecka et al., this volume). A major discontinuity is that only humans develop a concept of natural number. As children acquire the verbal count list, it provides an avenue for distinguishing between sets that cannot be accommodated by the ANS. In other words, the ANS cannot support our ability to distinguish between 18 and 19 apples, however by verbally counting we can easily do so. Without the verbal count list and the successor principle, full-blown mathematics as observed in human societies would not be possible (Carey 2009; Spelke 2011).

At first blush, the cases reviewed by Beran (this volume) in which animals appear to have learned a mapping between numerosities and arbitrary symbols may suggest that there is more continuity than discontinuity between animal and human minds in the domain of numerical cognition. However, we argue that there remains a fundamental difference that sets all nonhuman animals apart from human children. As reviewed by Sarnecka &

Goldman (this volume) somewhere around the age of four years children internalize the successor principle. They grasp that verbal counting is a means by which they can determine the cardinality of a set. When they do this it allows a mapping that transcends arduous learning of each symbol-referent pairing and lays the foundation for uniquely human mathematics (Carey 2009). In contrast, the few studies that have trained animals to match a symbol to a numerosity reveal no evidence that the animals show a learning curve or make an inference such as 'this new symbol represents the previous numerosity plus one'. Instead non-human animals seem to learn each new pairing as a new problem. Furthermore there is no compelling evidence that acquisition of numerical symbols allows nonhuman animals to represent large values precisely (although see Pepperberg & Carey 2012 for a different position). Most importantly, while there are many examples of animals attending to numerosity spontaneously it is inarguable that no species other than humans has independently devised systems for representing number symbolically.

How are the evolutionary and developmental building blocks of numerical cognition transformed into uniquely human mathematical cognition? We have discussed two systems for representing number without language. The ANS which is ratio-dependent and can represent number independently from other stimulus attributes (Feigenson et al. 2004), and the object-file system which is almost accidently numerical and is potentially more bound to continuous variables (Feigenson et al. 2004). An important question for this field is whether either of these systems provides the conceptual basis for symbolic math? As children learn the meaning of number words and symbols they must ground these symbols with nonverbal representations. What work does each system do in this important process?

Sarnecka & Goldman give an account of how children come to have natural number concepts. They offer a compelling account of the 'conceptual-role bootstrapping' hypothesis proposed by Carey (2009) which maintains that natural number concepts are constructed. The procedural counting routine that children master at an early age serves as a placeholder structure whereby the symbols are ordered but do not yet represent natural numbers for the child. Over time the symbols take on more meaning and become fully defined natural number concepts. Under this scenario the ANS is not part of this construction but is mapped onto natural number concepts later in development only after the child has become a cardinal principal knower and has internalized the successor principal. However as reviewed by Gilmore (this volume) and Gebuis & Reynvoet (this volume) recent data linking individual differences in the acuity of the ANS with individual differences in symbolic math ability suggest that the ANS may play a more important role than the bootstrapping hypothesis allows. There is agreement that children have mapped symbols onto ANS representations by the age of five, as suggested by the distance effect when they compare Arabic numerals, (Duncan & McFarland 1980; Sekuler & Mierkiewicz 1977), however the specific role if any the ANS plays in grounding children's symbolic understanding of number is not understood.

The quest for the cognitive foundations of mathematics

As reviewed by Gilmore (this volume) and Gebuis & Reynvoet (this volume) there are a growing number of recent datasets that demonstrate that some of the variance in symbolic math performance, as measured by SATs or standardized math achievement tests, can be explained by ANS precision as measured by Weber fraction (DeWind & Brannon 2012; Halberda & Feigenson 2008, Halberda et al. 2012; Libertus et al. 2011, 2012; Lyons & Beilock 2011; Piazza et al. 2010; Gilmore et al. 2010; Mazzocco et al. 2011). However other studies have failed to find any relationship between ANS and symbolic math achievement (Sasanguie et al. 2013; Holloway & Ansari 2009; Inglis et al. 2011; Iuculano et al. 2008; Nosworthy et al. 2013; Price et al. 2012; Soltesz et al. 2010; Wei et al. 2012), or have found that the relationship is mediated by symbolic number knowledge (Lyons & Beilock 2011), executive function (Fuhs and McNeil 2013), or only holds for children with low math ability (Bonny and Lourenco 2013).

An alternative possibility, as reviewed in Gebuis & Reynvoet, (this volume), is that facility with symbol ordering may play a more critical role in mathematical cognition and competence. For example, facility with ordering Arabic numerals is predictive of group differences between those with and without math deficiencies (Rousselle & Noel 2007; De Smedt & Gilmore 2011; Iuculano et al. 2008) and explains individual differences in typically developing children and healthy adults (Castronovo & Gobel 2012; Holloway & Ansari 2009; Bugden & Ansari 2011; Sasanguie et al. 2012; De Smedt et al. 2009; Landerl et al. 2004; Lyons & Beilock 2011). Clearly a great deal of future research is needed to uncover the mechanisms underlying the interactive links between ANS, symbolic number system, and math. Thus the relationship between ANS acuity and symbolic math may not be clear cut and further research is needed to determine whether it is mediated by other cognitive factors.

Most of the research exploring the relationship between the ANS and math achievement has been limited to a correlational approach, and very little work has yet to investigate the causal relationship between ANS and math. Two recent studies attempt to probe a possible causal relationship. We used a pre- and post-test training paradigm in adult participants and found that approximate arithmetic training improved symbolic math performance in two experiments (Park & Brannon 2013). Specifically participants were given six to ten days of training in which they solved approximate non-symbolic addition and subtraction problems. A second group of participants either received no intervening training or were trained on other tasks. Prior to and after the training sessions, all participants solved two- and three-digit symbolic arithmetic problems. The participants who were trained on non-symbolic approximate arithmetic showed stark improvement in symbolic math compared to participants in other groups. While the exact mechanisms underlying these transfer effects are not yet understood, these results suggest a promising avenue for future research and provide the first evidence that at least part of the relationship between ANS and math may indeed reflect a causal relationship. In another study, Hyde & colleagues (in press) asked first-grade children to complete an exact symbolic arithmetic test. Critically, prior to this test, some of the children received a small set of problems that engaged representation and operation of approximate numerical quantities while others received other kinds of non-numerical problems. Interestingly, those children who received the ANS-based problems prior to the symbolic arithmetic test performed better at the symbolic arithmetic test compared to other children. There are a few limitations of the study such as an absence of a pre- and post-test design, brief exposure of the training problems, and immediate testing, which restricts the interpretations of these results. Nevertheless, they support the hypothesis that there is a causal relationship between ANS and symbolic math and suggest the exciting possibility that training aspects of the ANS may be useful in improving symbolic mathematics in children.

If there is in fact a causal relationship between ANS and symbolic math this still leaves open the question of why? What role does ANS play in scaffolding symbolic math? On the one hand, ANS enables numerical quantity estimations and operations, so it is plausible to expect that formal mathematics, which involves the representations and operations of numbers, is rooted in the ANS. On the other hand, it is challenging to understand how such a primitive system that is not capable of representing exact large numbers could give rise to formal mathematics that is uniquely human.

One possibility is that young children with high ANS acuity have more distinct internal representation of large numerosity that may facilitate better symbol to numerosity mapping, which in turn may facilitate the learning of symbolic number system. This possibility is supported by recent findings indicating that children's ANS acuity measured prior to their formal math education at the age of three to four years correlates with their performance in standardized math achievement tests measured at the age of five to six (Mazzocco et al. 2011). Furthermore, six-month-old infants' ANS acuity measured from a change detection paradigm correlates with their standardized math scores three years later (Starr et al. 2013b). However, these findings do not explain why the relationship between ANS representations and math achievement appears to hold even into adulthood (e.g. Lyons & Beilock 2011; DeWind & Brannon 2012; Halberda et al. 2012). Thus another possibility is that ANS representations allow people to reject incorrect calculations that are off by orders of magnitude (Feigenson et al. 2013).

Yet a third possibility is that the causal arrow is reversed and that mathematical ability instead functions to sharpen ANS acuity. For example, children who are particularly adept at math or enjoy math will spend more time playing with numbers, which in turn may facilitate better performance in ANS tasks. In this way, the link between ANS and math may be bidirectional into adulthood. A recent study by Piazza & colleagues (2013) supports this idea in that indigenous Amazonian people with access to schooling have higher ANS acuity than those without access to schooling. Similarly, Hannula-Sormunen (this volume) argues that a domain-selective attentional system for number, namely the spontaneous focus on number, is another important factor that gives rise to mathematical competence in such an interactive way throughout development.

A final caveat is in order. Even if the answer turns out to be that ANS precision predicts some of the variance in math ability and that this reflects a causal relationship, it is notable that in most samples only a very small proportion of the variability in math competence appears to be explained by ANS acuity. Thus, even if there is a direct causal relationship between ANS and math there are likely many other more important predictors of math aptitude. Understanding the causal relationship may do more for the epistemology of numerical cognition than it does for practical interventions. In other words, understanding the relationship between the ANS and symbolic math is critical for uncovering the developmental roots of uniquely human numerical cognition; however, whether this link can be harnessed to meaningfully improve symbolic math is an open and distinct question.

Conclusions and outstanding questions

Why has the study of the evolution and development of number concepts received so much attention? A primary reason is that it provides one of the best examples of abstract thought. Even more so than categorical distinctions such as vehicle vs animal, number is abstract in that it can be divorced from perceptual variables. Three giraffes and three cars look, smell, feel, and taste nothing alike and yet they are both sets of three. The study of number therefore provides a case study for examining how complex cognition emerges over evolution and within the human lifespan. The chapters in this section illustrate the remarkable primitives that allow a wide range of species and prelinguistic humans to represent the quantitative aspect of the world within which they live. The chapters also raise compelling questions about how these primitives give rise to uniquely human mathematical cognition and should foster many questions for future research such as:

(1) What are the factors that determine whether animals and babies rely on the ANS or object-file system? Are there species differences in the presence or limits of the object-file system?
(2) What are the fundamental cognitive differences that allow the human mind to transcend the ANS that are lacking in non-human animals?
(3) How do children acquire natural numbers? Is there a causal role for both object-files and ANS in this process?
(4) What is the relationship between the ANS and symbolic math throughout the lifespan? What are the causal mechanisms that drive this relationship?
(5) How can the knowledge we gain about the primitives of numerical cognition be used to improve school-based math?

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