1. Classical electron distribution function: let’s derive a distribution function for classical particles. What makes a particle “classical” instead of “quantum” is the distinguishability (or lack of indistinguishability)—a “classical” object, such as any macroscopic object, can be somehow “tagged” and distinguished from all other objects of the same kind. Hence, when we put classical particles in energy “bins”, we can assume they are all different so we have $g_i$ choices for each of the $n_i$ particles. However, we have to also account for the fact that there are many different ways now to choose a specific set of $n_i$ particles out of $N_{tot}$ to place them into the $i$'th level. We end up (as did Ludwig Boltzmann more than a century ago) with the number of different microstates (arrangements) of $n_i$ particles into $g_i$ bins in the $i$'th level as

$$W_i = \frac{g_i^{n_i}}{n_i!}$$

a. Following the derivation in class, write down the total number of microstates $W$ based on the $W_i$ above.

b. Write the function $f(n_i)$ to be minimized by taking the entropy $S=k_B \ln W$ and adding in constraints $\alpha$ for the total number of particles, and $\beta$ for the total energy.

c. Simplify $f(n_i)$ by taking the natural logarithm of the product and using a slightly more accurate version of Stirling’s rule: $\ln(x!) \approx x \ln(x) - x$

d. Take the partial derivative of $f(n_i)$ with respect to $n_i$ and set it equal to zero

$$\frac{\partial f(n_i)}{\partial n_i} = 0$$

e. Solve for the ratio $n_i/g_i$ after rearranging and exponentiating the expression you derived in d). Do you recognize this distribution function (after the usual substitutions $\alpha=-E_F/T$ and $\beta=1/T$)

2. Heat Capacity of classical particles: Let’s consider classical particles, which differ from bosons and fermions in their distribution function is given by the Maxwell-Boltzmann distribution

$$f(\omega, T) = \exp \left( -\frac{\hbar \omega}{k_B T} \right)$$

a) Calculate $C(T)$ assuming the 3-dimensional Debye model for the dispersion $\omega(k) = c_s |k|$ where the dispersion is linear in the wavevector.

b) What is $C(T)$ in the high temperature limit? What should it be and do they match?

c) What is $C(T)$ in the low temperature limit? What should it be and does your answer match the expected limit?

d) The other historically important model of vibrations is Einstein’s model using harmonic oscillators with a given fixed frequency $\omega_k$. Calculate $C(T)$ using Einstein’s model and the classical Maxwell-Boltzmann distribution function.

e) What is $C(T)$ in the high $T$ limit and does it match your expectation of what it should be?

f) What is $C(T)$ in the low $T$ limit and how does it compare to what it should be?
3. In deriving the heat capacity of electrons, we relied on the electron density being

\[ n_e = \int_0^\infty dE_{FD}(E, T) D(E) \]

a. Using the effective mass approximation and the corresponding DOS, write down the integral we need to perform to calculate the electron density.

b. The expression in a) cannot be integrated in closed form because of the Fermi Dirac distribution; however, we can avoid it altogether by going to the low temperature limit. What does the Fermi-Dirac distribution tend to in the low T limit?

c. Write down a simplified form of a) using your answer to b). Hint: you should only have \( D(E) \) in the integrand, and a new upper limit of integration. What is the highest energy an electron can occupy at T=0K?

d. Perform the integral in c) and express it in terms of \( E_F \), the Fermi level, and the density of states at the Fermi level \( D(E_F) \).