1. Graphene is a unique material made up of a single atomic layer of carbon atoms arranged in a honeycomb lattice, a small segment of which is drawn below on the left:

The lattice vectors (marked in red in the figure above) are \( \vec{a}_1 = \frac{a}{2} \left( 3, \sqrt{3} \right) \) and \( \vec{a}_2 = \frac{a}{2} \left( 3, -\sqrt{3} \right) \) where \( a = 1.42 \) Angstrom is NOT the lattice spacing but rather the bond length (distance between two neighboring atoms).

a) How many atoms per basis does graphene have?

b) What is the area of the unit cell? Hint: see c) below

c) What are the reciprocal lattice vectors? Hint: you can use the definitions for 3D by adding the 3rd dimension with a 3rd lattice vector \((0,0,1)\) in the \(z\) direction and add a zero to \(a_1\) and \(a_2\) in the \(z\) direction. Then compute \(b_1, b_2,\) and \(b_3\) as done in class and finally remove the \(z\) coordinate of \(b_1\) and \(b_2\) and ignore \(b_3\).

d) Draw the 1st Brillouin zone by bisecting the first few reciprocal vectors \( \vec{G} = \alpha \vec{b}_1 + \beta \vec{b}_2 \) connecting the center of the 1st Brillouin zone to its neighbors.

e) How many unique corners does each BZ have? Hint: count the number of corners then divide by the number of neighbors that share each corner.

f) These unique corners are called \(K\) points. Give their locations.

g) The electronic structure of graphene can be approximated by a pair of cones at each \(K\) point, which can be written as \( E(\vec{k}) = \pm \hbar v_f |\vec{k} - \vec{K}| \) (note the + sign for the top cone and the – sign for the bottom cone, highlighted in the figure on the right above). What is the density of states of graphene? Draw a picture of the DOS(E).

h) We introduced the group velocity as the gradient of the dispersion

\[
\nu_g(\vec{k}) = \nabla_k \omega(\vec{k}) = \frac{1}{\hbar} \nabla_k E(\vec{k})
\]

What is the group velocity of graphene using the \(E-k\) defined in the previous problem?
2. We want to find the density of states (DOS) of a 2-dimensional lattice in the effective mass approximation.

a) Write down the expression for \( E(k) \) in the effective mass approximation for a 2-d lattice. 
   Hint: it is the same as in 3D but the wavevector is 2-dimensional (there is only \( k_x \) and \( k_y \)).

b) What is the size of the constant energy contour in this 2-dimensional lattice (having also a 2-dimensional reciprocal space)? Reminder: the size of the constant energy contour in 3d was the surface of the constant energy sphere \( S(k) = 4\pi k^2 \) and \( S(E) = 8\pi m^* E/\hbar^2 \).

c) If your answer in b) was not in terms of energy already, then express it as a function of energy.

d) What is the gradient of \( E(k) \)? Express the gradient as a function of \( E \). Hint: should it be different in 2d form what it was in 3d?

e) Put your answers for c) and d) together to give the DOS \( g(E) \) (don’t forget that in 2-d the prefactor should be \( 1/(2\pi)^2 \) instead of \( 1/(2\pi)^3 \)).

3. Derive an expression for the density of states of a general isotropic dispersion where the constant energy surface is a sphere and the \( E-k \) relationship is given as a power of the magnitude of the wavevector as \( E\left(\vec{k}\right) = \alpha |\vec{k}|^\beta \). Note that the examples we did in class were for \( \beta = 1 \) and \( \beta = 2 \), linear for phonons and quadratic for electrons.