PLENITUDE OF POSSIBLE STRUCTURES

Our chief concern is with actuality, with the way the world is. But inquiry into the actual may lead even to the farthest reaches of the possible. For example, to know what consequences follow from a supposition, we need to know what possibilities the supposition comprehends. Suppose that space is unbounded; does it follow that space is infinite, as was once generally believed? The possibility of "curved" space demonstrates the opposite. Inquiry is driven by logic, and logical relations hold or fail to hold according to what is logically possible.

Whence come our beliefs about logical possibility? Typically, they derive from the analysis of non-logical concepts, general and individual. I know that no bachelor could possibly be married because of what I know about necessary and sufficient conditions for the application of the concept of bachelor. I know that Ronald Reagan could not have discovered the theory of relativity, if I do, because of what I know about necessary and sufficient conditions for being the individual Ronald Reagan. But not all beliefs about logical possibility can be attributed to the analysis of non-logical concepts. Sometimes we reason in accordance with general principles that are constitutive of the concept of logical possibility itself, principles to the effect that, if such-and-such is possible, then such-and-so must be possible as well. Call these <u>principles of plenitude</u>. I divide them into three sorts. First, there are principles that require a plenitude of recombinations. We reason according to such principles, for example, when we argue that it is logically possible for there to be a human head attached to the body of a horse. Second, and more controversially, there are principles that require a plenitude of possible contents. We reason according to such principles, for example, when we argue that some or all of the actual individuals and properties could be replaced by individuals and properties not of this world: alien individuals and properties. Finally, there are principles that require a plenitude of possible structures. We reason according to such principles, for example, when we argue that, if it is logically possible for there to be four or five spatial dimensions, then it is logically possible for there to be seventeen, or seventeen thousand. These three sorts of plenitude, taken together, delimit the scope of the possible.

In this paper I consider only the last-mentioned plenitude: plenitude of possible structures. I take it that there are structures that we know to be logically possible, for example, the three-dimensional Euclidean and non-Euclidean spaces of constant curvature. My goal is to uncover the source of that knowledge, and, in so doing, to combat skepticism about modality without appealing to any mysterious faculty of modal intuition. On my account, our knowledge starts from our theorizing about the actual world, and is extended, in accordance with the demands of plenitude, by the results of mathematics. I develop and defend a principle of plenitude for structures, and motivate the principle pragmatically by way of the role that logical possibility plays in our inquiry into the world. Along the way, I compare my account with views put forth by Robert Adams and David Lewis.

First, some preliminary points. A structure is logically possible, on my usage, only if there are or could be concrete entities that instantiate that structure, that is, only if the structure is instantiated by (some or all) of the concrete inhabitants of some possible world.[1] A structure is instantiated by a plurality of inhabitants of a world in virtue of their natural properties and relations; otherwise, structures would be too easily instantiated, since instantiation would depend only upon cardinality.[2] I assume that structures exist as abstract entities of some sort.[3] If something more definite is wanted, structures may be represented set-

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[1]Two points. (1) Some authors use 'metaphysical possibility' for what I call 'logical possibility', and reserve 'logical possibility' for some (prima facie) weaker notion of "mathematical" (or "conceptual") possibility: a putative abstract entity is mathematically possible, roughly, if it can consistently be posited to exist. All structures are possible in this sense. (2) Although the inference from possibly instantiated to instantiated at some possible world is sometimes controversial (e.g., when applied to alien properties), it does not seem problematic when applied to structures; at any rate, I will assume that 'world' is taken in a broad enough sense so as to make this so.

[2]Not every class of entities at a world is the extension of a natural property; belonging to the extension of a natural property may be a matter of shared universals, or duplicate tropes, or primitive naturalness applied to classes of possibilia; I need not decide that here. For discussion and comparison of these views, see David Lewis, On the Plurality of Worlds (Oxford: Basil Blackwell, 1986), pp. 59-69. I discuss naturalness more extensively below.

[3]Platonism about structures could be avoided by taking "mathematical possibility" as a primitive, and speaking of structures as mathematically possible rather than abstractly existing. The problem of plenitude of structures would then be the problem of determining the relation between mathematical possibility and logical possibility. Nothing that follows depends upon this choice.

theoretically in ways familiar from model theory for first-order languages.[4]

I will focus in what follows upon spatial and spatiotemporal structures. Not because I think these are the only sorts of structure to which plenitude applies: worlds have a pattern of instantiation of non-spatiotemporal natural properties and relations; and perhaps some worlds have irreducible causal, or nomological, or probabilistic structure. I focus upon spatial and spatiotemporal structures because they provide substantive examples upon which there is at least some initial agreement as to possibility.[5]

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[4]For example, in the simplest case, we can take a <u>model</u> to be an ordered pair whose first member is a set, called the <u>domain</u> of the model, and whose second member is a set of properties or relations (set-theoretically construed) on the domain. Each model represents a unique structure; isomorphic models represent the <u>same</u> structure. A structure is <u>instantiated</u> at a world if and only if it is represented by some model whose domain consists of (concrete) entities existing at the world, and whose second member consists of extensions of natural properties and relations.

[5]Beware. I normally use 'space' and 'spatial structure' interchangeably to refer to a "mathematical" entity; but 'space' also has a physical interpretation. Thus, when I say that Euclidean space is instantiated at a world, I do not thereby say that physical space at the world is Euclidean. The latter requires that the structure, Euclidean space, be instantiated by the right entities (all the points of physical space, for the realist), and perhaps also in virtue of the right natural relations (for example, the distance relation at the world).

There is one more piece of business before turning to plenitude of structures. A principle of plenitude for structures, on my account, does not by itself determine which structures are possible. It serves rather as a principle of inference for modal reasoning: given that these initial structures are possible, these other structures are possible as well. The possibility of the initial structures must be believed on independent grounds. What might these be?

Consider Newtonian spacetime: any two events have an absolute spatial and an absolute temporal separation. I assume we all believe that Newtonian spacetime is logically possible. But, thanks to Einstein, we no longer believe it is actual, or even compatible with the actual laws. This suggests that logical possibility is required to encompass, not only actuality and nomological possibility, but our theorizing about actuality and nomological possibility as well. I propose:

(B) We have warranted belief that a structure is logically possible if that structure plays, or has played, an explanatory role in our theorizing about the actual world.

A number of comments are in order. (1) Condition (B) makes warranted belief about logical possibility relative to history and to a community of theorizers, as it should; it does not make logical possibility itself relative. (2) As the case of Newtonian spacetime suggests, the historical relativity is asymmetric: the structures believed with warrant to be possible by a community only increase over time. (3) If a bad theory takes hold in a community, positing gratuitous structure that explains nothing, condition (B) does not apply. To "play an explanatory role" is not just to be taken to play an explanatory role by the community. The structure must have genuine explanatory power.[6] (4) Although it is primarily scientific theorizing about the world that I have in mind, (B) is not thus restricted. Philosophers and theologians have posited explanatory structures in their theorizing about the world. However much we mistrust their speculations, we should not exclude these structures without cause. We can eliminate bad philosophy or theology in the same way we eliminate bad science: by requiring genuine explanatory power. (5) It is enough for a theory to be seriously considered by a community; it need not ever be believed. Belief in the possibility of Lobachevskian space (of very small negative curvature) is warranted by (B), because it was seriously considered (in the nineteenth century) whether measurements of stellar parallax supported the Euclidean or Lobachevskian theory of space. (6) Whenever a structure is instantiated at a world, so are all its substructures. For example, a world at which three-dimensional Euclidean space is instantiated is also a world at which one- and two-dimensional Euclidean space is instantiated. Thus,

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[6]Perhaps even Newtonian spacetime fails this test due to its gratuitous positing of absolute rest; in which case only so-called Neo-Newtonian, or Galilean, spacetime, which posits absolute acceleration but not absolute rest, could be warranted by (B). The possibility of Newtonian spacetime would then be derived from plenitude. For the distinction between Newtonian and Galilean spacetime, and a discussion of the explanatory adequacy of spatiotemporal structures, see Michael Friedman, Foundations of Space-Time Theories (Princeton: Princeton University Press, 1983), pp. 71-92, 236-263.

warranted belief in the possibility of a structure passes to all of its substructures. For convenience, I will interpret "plays an explanatory role in our theorizing" in such a way that, whenever a structure plays such a role, all of its substructures do so as well.

II

At last, I turn to plenitude. The structures satisfying (B) are not the only structures we believe to be possible. There are structures we believe possible that neither play, nor have played, any explanatory role in our theorizing. We believe them possible, I suppose, because we believe that the space of logical possibilities must be "filled out" or "completed" in some non-arbitrary way. But what counts as arbitrary here? Can these constraints on logical space be made more precise?

As a first try, we might take the intuitive idea underlying plenitude to be that "there are no gaps in logical space."[7] But what constitutes a gap? Suppose that there are worlds with Euclidean space of six dimensions, and worlds with Euclidean space of eight dimensions, but none with Euclidean space of seven dimensions. Would that be a violation of plenitude, a gap in logical space?

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[7]From David Lewis, On the Plurality of Worlds, p. 86. Other expressions Lewis uses for the intuitive idea of plenitude include: "the worlds are abundant"; "logical space is somehow complete"; [there are] no vacancies where a world might have been but isn't" (p. 86).

It would indeed; but one must be cautious in giving the reason. Six-sided regular polyhedra (cubes) are logically possible, as are eight-sided regular polyhedra (octohedra), but not seven-sided regular polyhedra. Yet that does not constitute a gap in logical space. Wherein lies the difference? There _is_ a gap in the first case because mathematical generalizations of three-dimensional Euclidean space to higher dimensions include a seven-dimensional space whenever they include six- and eight-dimensional spaces; and they provide a natural ordering of the spaces according to which the seven-dimensional space falls between the other two. There is _no_ gap in the second case because mathematics teaches us that a seven-sided regular polyhedron is a contradiction in terms; so in going from six-sided to eight-sided, nothing has been left out. In sum, mathematics provides the backdrop of structures and the natural orderings on structures, without which the notion of a gap in logical space would make no sense.

It is not enough, however, to rule out gaps in logical space; plenitude demands that logical space contain no arbitrary or unnatural boundaries. Suppose that Euclidean spaces of all dimensions up to six were logically possible, but none of greater dimension. That too would be a violation of plenitude. The mathematical generalization of three-dimensional Euclidean space to four-, five-, and six-dimensional Euclidean space applies, _mutatis mutandis_, to all finite dimensions; there is no natural stopping point among the finite-dimensional spaces. To allow that some but not all finite-dimensional Euclidean spaces are logically possible would assign an unnatural boundary to logical space.

The idea that logical space contains no unnatural boundaries can be taken to supercede and clarify the idea that it contains no gaps. A gap in logical space is formed by two boundaries, one from either side. Call a gap natural if both its boundaries are natural; unnatural otherwise. A prohibition on unnatural boundaries entails a prohibition on unnatural gaps; natural gaps in logical space, if any there be, need not be a violation of plenitude.

It should be apparent by now that an account of plenitude must rely heavily on a notion of naturalness (or some equivalent). I will assume that naturalness applies to classes generally, and, in particular, to classes of structures. Talk of natural boundaries in logical space is easily translated into talk of natural classes: any class of logically possible structures determines a boundary in logical space; the boundary is natural just in case the class is natural, or is a union of natural classes. Although I have no analysis of naturalness to offer, some words of clarification and illustration are in order.

Naturalness applies both to classes of physical entities and to classes of mathematical entities. In either case, what the natural classes are is not determined by us: it is a matter of objective, non-contingent fact. Examples of natural classes of mathematical entities include: the natural numbers, the real numbers, the ordinal numbers, recursive functions of natural numbers, continuous functions of real numbers. Examples of natural classes of mathematical structures include: groups, vector spaces, topological spaces, Euclidean spaces. Each of these natural classes serves as the principle object of study for some major area of mathematics. If a working criterion for naturalness is wanted, we have here, at least, a sufficient condition. That is not to say, however, that the above-mentioned classes are natural because mathematicians have chosen to study them. Rather, mathematicians have chosen to study them, I take it, in part because they are natural classes.

Natural classes arise in mathematics in two complementary ways: by postulate and by construction. The class of structures satisfying some natural set of postulates is, I suppose, a natural class; here one finds groups, lattices, and other structures familiar from abstract algebra. On the other hand, natural classes of structures may be constructed from a given class of structures by some natural operation on (classes of) structures. The constructions that will be of primary interest to us are the mathematically natural processes of generalization.

Although I will speak of classes simply as natural or unnatural, it is clear that naturalness is a matter of degree. The odd natural numbers do not form a natural class in the sense here intended: the study of odd number theory, as opposed to number theory, would be a largely fruitless endeavor. But the odd numbers deviate from naturalness less than the numbers that are odd up to a hundred and even thereafter; and these numbers in turn deviate from naturalness less than some really gruesome class of numbers not even definable within elementary arithmetic. For what follows, I need to assume that classes of structures may be <u>perfectly</u> natural, that there is a greatest degree of naturalness; when I say a class is 'natural', I mean 'perfectly natural'. Perhaps that assumption is controversial; in any case, I will not try to defend it here.

Since structures can themselves be represented by classes, they can be judged natural or unnatural, one by one, according to the naturalness of their representatives. It is important, however, not to equate the naturalness of a <u>class</u> of structures with the naturalness of its members. For example, the class of partial orders is a natural class of structures, even though it contains some gruesome members that no one would (or could) consider individually. In the other direction, a disparate collection of individually natural structures need not form a natural class. But note one exception: the naturalness of a singleton structure, I take it, goes by the naturalness of its sole member.

Naturalness itself imposes a structure on the classes of structures. Some assumptions about this structure will be needed below. I assume that the natural classes exhaust the class of all structures, that is, that every structure belongs to some natural class. I assume that the natural classes are not closed under unions or complements. Perhaps they are closed under intersections, but since that is controversial, I will not assume it in what follows. Finally, I assume that the class of all structures is <u>not</u> a natural class, on grounds of heterogeneity; but the formulations below could easily be revised to accommodate the contrary judgment.

III

With the notion of naturalness of classes in hand, I turn to formulations of a principle of plenitude for structures. The easiest way to meet the demand that there be no unnatural boundaries is to draw no boundaries at all:

(P1) Every structure is a logically possible structure.

I find (P1) attractive as a principle of plenitude for structures. For one thing, it provides an exceedingly simple account. Once (P1) is accepted, (B) becomes superfluous; mathematics alone--perhaps, mathematical logic alone--determines which structures are possible. Moreover, though the notion of naturalness may play a role in motivating (P1), it plays no role in its formulation. Unfortunately, (P1) goes far beyond anything demanded by the idea that logical space be characterizable in a non-arbitrary way.[8] Perhaps (P1) could be defended by way of the benefits it confers upon our total theory. In any case, I will here remain agnostic towards (P1), and go on to develop a (somewhat) more conservative principle that is capable of a stronger defense.

There is another simple way to meet the demand that the space of possible structures contain no unnatural boundaries:

(P2) The class of logically possible structures is a natural class.

(P2) constrains the shape of logical space. It does not by itself tell us whether any particular structure is logically possible. But when combined with (B), it may support inferences to the possibility of particular structures. Thus, let B be the class of structures warranted by (B). Any structure that belongs to every natural class of structures that includes B is warranted by (P2). (I say a structure is warranted, for short, if belief in its logical possibility is warranted.) For example, suppose that B contained only the Euclidean spaces of one-, two-, and three-dimensions; then (P2) would warrant the other finite-dimensional Euclidean spaces.

However, (P2) will not do as a principle of plenitude for structures: it is both too strong and too weak. To see that it is too strong, consider the

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[8]And, we shall see below, beyond what I take to motivate that idea: the role that logical possibility plays within our inquiry into the actual world.

class of logically possible spatiotemporal structures. I take it we believe, based upon (B), that this class includes both continuous and discrete spacetimes,[9] but I do not believe that any natural class encompasses them both; the mathematics of continuity and the mathematics of discreteness have little in common. Thus, B is not included in any natural class, making the acceptance of (P2) incompatible with (B).

A solution is not far to seek. Although the class of possible spacetimes is not a natural class, it is a union of natural classes; we call them all "spacetimes" not because they form a natural mathematical kind, but because of some looser family resemblance. This suggests that we weaken (P2) as follows:

(P3) The class of logically possible structures is a union of natural classes.

(P3) still constrains the shape of logical space, assuming, at any rate, that singletons are not in general natural classes. But (P3) is genuinely weaker than (P2) because the natural classes are not closed under unions. Moreover, when combined with (B), it still supports inferences to the possibility of particular structures: given a structure b in B, (P3) warrants any structure that belongs to every natural class containing b. Finally,

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[9](B) supports belief in the possibility of discrete spacetimes that are observationally indistinguishable from the continuous spacetimes posited by physical theory. The physical possibility of discrete space, time, or spacetime has been taken seriously by scientists and philosophers from antiquity to the present day.

(P3) is still sufficiently strong to guarantee that the space of possible structures contain no unnatural boundaries.

Nevertheless, I think (P3) is too weak in at least two ways. And if I am right, the condition that logical space contain no unnatural boundaries cannot be sufficient for plenitude. First, there is a problem of crosswise generalizations. Suppose that there are two natural ways of generalizing from a structure b in B, resulting in two natural classes containing b. If these generalizations cut crosswise, they may have only the structure b in common; in which case, no inference from the possibility of b to the possibility of any of the structures that generalize b will be supported by (P3). Consider this example. Suppose again that three-dimensional Euclidean space is one of the structures in B. One can generalize the number of dimensions to any finite value while keeping the space Euclidean, or generalize the curvature to any constant negative or positive value while keeping the space three-dimensional. Both generalizations, it seems to me, result in natural classes of spaces. It is compatible with (P3) that the spaces from only one of these classes be possible. But that is too weak. I think we have grounds to infer that all the spaces in question are possible, grounds that (P3) fails to capture. (P3) allows crosswise generalizations in effect to cancel each other out, without consequence.

One might simply concede that cross-generalizations on a single structure b cancel one another unless there are other structures in B that, together with b, support inferences to the structures that generalize b. Thus, plenitude of structures demands that all finite-dimensional Euclidean spaces be possible only because B contains, in addition to the three-dimensional Euclidean space, the one-, and two-dimensional Euclidean spaces; and any natural class containing these three spaces contains all finite-dimensional spaces. (Similarly, all three-dimensional spaces of constant curvature are possible because B contains three-dimensional spaces of (very small) negative and positive constant curvature.) This suggests it might suffice to enhance (P3) as follows:

> (P4) The class of logically possible structures is a union of natural classes. Moreover, suppose S is a class of logically possible structures that is included in some natural class. Any structure that belongs to every natural class of structures that includes S is logically possible.

(P4) falls midway in strength between (P2) and (P3): unlike (P3), it permits inferences from classes of structures, not just from single structures; but unlike (P2), it does not require that every class of logically possible structures be included in some natural class.

Is (P4) strong enough to capture plenitude of structures? I think not. For (P4) as well as (P3), there is a problem of nested generalizations. Consider the supposition that there are possible Euclidean spaces with any finite number of dimensions, but no possible Euclidean spaces with infinitely many dimensions. This supposition posits no unnatural boundaries in logical space: the class of finite-dimensional Euclidean spaces is a natural class, an appropriate object of study in mathematics. Thus, the supposition violates neither (P2), (P3), nor (P4). But I claim it is a violation of plenitude nonetheless. The natural generalization of one-, two-, and three-dimensional Euclidean space to other finite dimensions can itself be naturally extended into the infinite. For example, there is a natural generalization of the Euclidean metric to spaces of continuummany dimensions which makes use of the way that integration generalizes finite summation.[10] Assuming that the Euclidean spaces in B are all finite-dimensional, it follows that they are included in at least two natural classes, one a subclass of the other. (P4) provides no grounds for inferring that any space contained only in the larger of the two subclasses--that is, any infinite-dimensional Euclidean space--is logically possible. But on what grounds does plenitude differentiate between the possibility, say, of a seventeen-dimensional Euclidean space, and the possibility of an infinite-dimensional Euclidean space? What does the size of a spatial structure have to do with the possibility of its instantiation?

One might reply: the seventeen-dimensional space is closer to the spaces in B than any infinite-dimensional space, according to the natural ordering of structures. But this reply is incompatible, at least in spirit, with the all-or-nothing approach to logical possibility taken by (P2) through (P4). If a relation of closeness to the structures in B is what differentiates the finite- and infinite-dimensional spaces with respect to possibility, it becomes an utter mystery why a space of seventeen thousand dimensions should be no less possible than a space of seventeen. The reply in question leads inevitably, I think, to the view that logical possibility is a matter of degree, in which case logical implication becomes a matter of degree as well. That is a truly radical view; I do not reject it out of hand, but it will not be considered further in this paper.

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[10]A standard example. Let the points of the space be the continuous real-valued functions defined on the real interval [0, 1]. Define the distance between two points, f and g, to be: $(g(x) - f(x))^2 dx$.

I know of no other grounds for favoring the finite-dimensional over the infinite-dimensional Euclidean spaces. I conclude that any principle of plenitude that warrants belief in the possibility of the former must warrant belief in the possibility of the latter. (P4) fails this test.

The same conclusion can be reached by a slightly different route. Suppose again that plenitude requires that there be no arbitrariness in logical space. One way for logical space to be arbitrary, I have said, is to have an unnatural boundary, that is, to not be a union of natural classes. But there is another way. Consider a nested sequence of natural classes representing more and more high-powered generalizations of some structures in B; suppose that any member of B occurs in the first member of the sequence or in no member at all; suppose further that any natural class that includes every class in the sequence is itself a member of the sequence. If Z is the union of all classes in the sequence, then Z contains all the structures that are candidates for logical possibility in virtue of the mathematical generalizations of the structures in question in B. Now, (P4) permits any division of Z into possible and not possible, so long as the possible structures form a natural class (and include the given structures in B). But it would be arbitrary for the boundary of logical space to follow one such division over any other. The only way to avoid such arbitrariness in logical space is to impose no division of Z. This suggests the following principle of plenitude:

(P5) Suppose s is a logically possible structure. Any structure that belongs to any natural class of structures containing s is logically possible.

(P5) substantially strengthens (P4). When combined with (B), it supports inferences to the possibility of spaces of any infinite dimensionality, as long as those spaces arise from a natural mathematical generalization of ordinary Euclidean space.

I wish I could in good conscience stop here; but a complication remains. There is a problem of overhasty generalization. Consider one-dimensional Euclidean space; that is, the structure of the real numbers with the usual metric: $\text{distance}(\underline{x}, \underline{y}) = |\underline{x} - \underline{y}|$. Is there any natural process of generalization that, when given only this structure as input, gives the finite-dimensional Euclidean spaces as output? I think not. The fundamental form of the Euclidean metric--being the square root of a sum of squares--plays no role in the one-dimensional case. Granted, one-dimensional Euclidean space is a special case of finite-dimensional Euclidean space; but it is too trivial a special case to support a generalization to higher dimensions. This leads to a problem with (P5). Given the possibility of only the one-dimensional Euclidean space, (P5) supports the inference to the possibility of all the finite-dimensional Euclidean spaces. That inference seems just as overhasty as the generalization upon which it is based.

There is an easy fix that should be resisted. We could say that plenitude of structures only supports inferences based upon generalizations involving two or more structures. But that fails to get to the heart of the problem. Natural generalizations can, I think, be based upon a single structure if that structure isn't a trivial or degenerate case of the generalization; perhaps three-dimensional Euclidean space is an example. On the other hand, two structures may be no better than one, if both structures are trivial cases of the generalization in question. The number of structures needed to support a generalization is relative both to the type of generalization and to the particular structures chosen; it cannot be specified, once and for all, in advance.

I see no choice, then, but to conclude that the notion of natural class is not by itself sufficient for formulating a principle of plenitude for structures; we need a relation that holds between a class of structures and those classes of structures that are natural generalizations of it. A natural generalization of a class of structures is always a natural class; but a natural class need not be a natural generalization of all of its subclasses. Switching from natural classes to natural generalizations transforms (P5) into:

(PS) Plenitude of Structures. Suppose S is a class of logically possible structures. Any structure belonging to any natural generalization of S is logically possible.

This is the principle of plenitude for structures that I accept. It shares all the virtues of (P5): the logical space of possible structures has no unnatural boundaries, nor arbitrariness in the way boundaries are set. Indeed, it may be that when applied to (B), (P5) and (PS) differ not at all with respect to the structures they warrant. But if and when they do differ, I stand by (PS).

IV

Thus far I have assumed without argument that logical space should have natural boundaries set in a non-arbitrary way. Can this assumption itself be defended? I think it can. I take it to be constitutive of logical possibility that it provide a suitable framework for our inquiry into the actual world; whoever denied this could not mean what I do by 'logical possibility'. Our inquiry into the actual world involves concepts--such as space, time, and spacetime--that have meaningful application beyond the actual world, indeed, beyond the nomologically and the doxastically possible worlds. Since part of that inquiry is inquiry into the nature of these concepts and their logical interrelations, logical possibility must extend at least as far as the meaningful application of these concepts.

Consider the question with which I began this paper: if (physical) space is unbounded, must it also be infinite in extent? Suppose the question had been asked in the 18th century, prior to the discovery of non-Euclidean geometry. I think the answer would have been "no" even then: 'space' did not then mean 'Euclidean space', any more than it does now. Thus, questions about the world that might well have been asked in the 18th century could only have been answered in the light of mathematical generalizations that were then unknown. The situation is no different today. We do not know in advance which mathematical generalizations of our concepts will turn out to be relevant to our inquiry.[11] If the class of logically possible structures includes some but not all of these generalizations, as is allowed by (P2) through (P4), then logical possibility may be unfit to provide a logical framework for our

[11]Actually, I hold something stronger, that we know in advance that every generalization is logically relevant, so long as it is compatible with whatever necessary conditions we place on the concept. But that depends upon a theory of content for concepts that I won't defend here.

inquiry into the world. In order to ensure that no relevant structure is left out of logical space, we need to posit a plenitude of possible structures, we need the space of possible structures to be filled-out in a non-arbitrary way.

The role that logical possibility plays in inquiry can motivate and justify both (B) and (PS); does it also support (P1), that every structure is logically possible? No; logical possibility must be broad enough to accommodate inquiry into matters of <u>contingent</u> truth, not matters of <u>necessary</u> truth. I do not require, nor is it customary to require, that logical possibility provide a framework for mathematics. If a structure does not belong to any mathematical generalization of any actual structure, or of any structure warranted by (B), then it is logically irrelevant to our inquiry into the actual world.[12] It could safely be excluded from logical space.

[12]Of course, it may be psychologically relevant by suggesting analogies, serving as a heuristic tool, and so on.

V

I wish now to illustrate my account of plenitude of structures, and to compare it with what others have said. I begin with an application of (PS) that has played an important role in the history of science: the development of Riemannian geometry. Consider the following questions. Is there a possible space with variable curvature, sometimes negative, sometimes positive? Is there a possible non-orientable space in which a right-handed glove could be made to coincide with a left-handed glove by transporting it to and from some distant place? I think it would be a mistake to claim ignorance here. These questions can be answered, and answered decisively, using (PS). Moreover, the answers do not come from some special faculty of modal intuition; according to (PS), they come straight from mathematics. Great advances in mathematics are often as much a matter of discovering what the natural classes are (the definitions) as discovering truths about them (the proofs). Indeed, it is the former sort of discovery that brings whole fields of mathematics into being. Riemann was the first to characterize the natural class of structures--the class of Riemannian spaces--that provides the objects of study in differential geometry. The way in which Riemannian space generalizes Euclidean space is difficult to make mathematically precise, but the underlying idea is simple enough: Riemannian space is <u>locally</u> Euclidean; it approximates Euclidean space in the small. The generalization is universally recognized to be natural, and to result in a natural class of structures. Thus, by (PS), all Riemannian spaces are possible structures (assuming Euclidean space is), and the answer to the above questions is "yes." It is because we accept (PS) that, as mathematics discovers more inclusive natural classes of structures, our beliefs about possibility expand accordingly.

So say I. Robert Adams has discussed this case, and come to a contrary conclusion.[13] Adams is concerned to argue against the idea, traceable to Leibniz, that there is a general presumption in favor of possibility in the absence of proof to the contrary. When applied to structures, this amounts to the claim that a structure should be presumed to be possible (i.e., possibly instantiated) unless and until its instantiation is proved to be impossible. (PS) does not embody any such rule of presumption; indeed, its role is to provide standards of proof for possibility, not to prescribe what to do in the absence of proof. But it is clear that Adams would hold that (PS), no less than a rule of presumption, is based upon "an unreasonable prejudice ... in favor of enlarging the extent of possibility." (P. 28). Discussing the case of Riemannian space[14], he writes:

> Should it have been believed metaphysically possible [in the 19th century] for there to be, for example, a "curved," Riemannian

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[13]In "Presumption and the Necessary Existence of God," Nous, 22, (1988), 19-32.

[14]Unfortunately, 'Riemannian space' in ordinary usage is ambiguous between (1) the spaces of constant positive curvature satisfying the axioms of elliptical geometry, and (2) the more general spaces of variable curvature studied in differential geometry. The mention of postulates in the following quotation suggests that Adams has the former in mind; the mention of current theories in physics suggests the latter. I always mean the latter by 'Riemannian space'.

> physical space that would satisfy the theses of Riemannian geometry? The metaphysical possibility of such a space does not follow from the formal consistency of the axiom and postulate set. In the absence of proof, should the possibility of curved physical space have been accepted on the ground that there is a presumption in favor of possibility? I think that would have been an implausible way of deciding the issue. ... If the majority opinion today is that curved space is metaphysically possible, the principal reason for this belief is surely not a presumption of possibility, but the fact that the actuality (and hence the possibility) of Riemannian space is implied by otherwise attractive theories in physics. This kind of broader theoretical consideration seems an eminently reasonable basis for deciding issues about metaphysical possibility. (Pp. 29-30)

This suggests that, before Einstein, we did not have warranted belief in the possibility of Riemannian space. Moreover, the only sorts of consideration that could warrant belief in the possibility of Riemannian space are (some of) those embodied in (B), not (PS). Adams seems to reject what I call plenitude of structures altogether.

Why does Adams think that prior to Einstein we should have been agnostic about the possibility of non-Euclidean space? He mentions here Kantian views according to which we can "just 'see', intuitively, that space <u>must</u> be Euclidean." (P. 29). Adams does not endorse such Kantian views; but he nonetheless holds that "the history of the subject" casts doubt on the claim that "if there were something impossible about non-Euclidean space we would probably have discovered it." (P. 30).

Although I agree with Adams' rejection of a general rule of presumption, his account of our belief in the possibility of non-Euclidean space does not seem to me cogent. For one thing, the Kantian views all rest on idealist suppositions to the effect that our mental faculties somehow constrain the structure of physical space; and I, for one, place no credence in that. For another thing, I do not think the history of the subject supports Adams' account. The view that we have a priori knowledge of the Euclidean nature of (physical) space was abandoned by most philosophers and mathematicians in the 19th century in direct response to the mathematical development of non-Euclidean geometry.[15] If Adams were right, it should not have been abandoned until the 20th-century development of general relativity; for, according to Adams, it was not until then that we had reason to believe that non-Euclidean space was possible. Moreover, suppose we concede that there is some broadly philosophical reason for withholding belief in the possibility of non-Euclidean space, a reason that was not undercut by the mathematical discovery of non-Euclidean geometry. I do not see how the development of general relativity could have undercut that reason. Einstein did nothing

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[15]To pick just one example, Riemann himself writes: "These facts [of Euclidean geometry] are ... not necessary but of a merely empirical certainty; they are hypotheses; one may therefore inquire into their probability, which is truly very great within the bounds of observation, and thereafter decide concerning the admissibility of protracting them outside of the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small." From "On the Hypotheses which Lie at the Foundations of Geometry," reprinted in David Eugene Smith, A Source Book in Mathematics (New York: Dover, 1959), p. 412.

to address the philosophical issues that, according to Adams, might be responsible for an as yet undiscovered impossibility in non-Euclidean space. So it is unclear, on Adams' account, why we should not remain agnostic about the possibility of non-Euclidean space, even in the light of current physics.

Finally, consider Adams' claim that the reason we believe today that Riemannian space is possible is that "the actuality (and hence the possibility) of Riemannian space is implied by otherwise attractive theories in physics." But which Riemannian spaces should we believe possible? Current theories of gravitation are compatible with some Riemannian spaces, but not all; indeed, even the spaces of constant positive and negative curvature are incompatible with current theory. Does Adams think we have reason to believe in the possibility of these simple non-Euclidean spaces? If not, then I submit that his view of possibility is too narrow, too far removed from the accepted conception. If so, then the mathematical relations between these simple spaces and the spaces that are compatible with current theory must play a supporting role. Adams must implicitly be supplementing (his version of) (B) with some principle of plenitude: from the possibility of some Riemannian spaces, the possibility of other Riemannian spaces follows. Of course, Adams might only endorse a principle much weaker than (PS). I have attempted to defend the full strength of (PS) above.

VI

I wish now to consider a second application of (PS). Unlike the previous case, it involves a process of generalization that is mathematically easy to specify. The resulting class of structures is natural, not because it is a mathematically interesting object of study, but because it is formed by a process of generalization universally recognized to result in natural classes. Not all natural classes of structures are mathematically interesting.

The process of generalization is this. Suppose that we are interested in some kind of structure, say, vector spaces. And suppose that particular structures of the kind in question have been characterized by specifying values for one or more independent parameters. For example, by specifying that the dimension is to be one, two, or three and that the vector components are to be real numbers, we uniquely characterize the vector spaces $V_1(\mathbf{R})$, $V_2(\mathbf{R})$, and $V_3(\mathbf{R})$. To generate a larger class of structures, it suffices to choose, for each parameter, a larger class of values. The resulting class of structures is a natural generalization of the original structures if and only if, for each parameter, the new class of values is a natural generalization of the old. Thus, for vector spaces, the number of dimensions might be taken to range over the natural numbers, and the source of vector components might be taken to range over arbitrary fields. The resulting class of vector spaces, $\{V_n(F)$: n a natural number, F a field$\}$, is a natural generalization of the vector spaces $V_1(\mathbf{R})$, $V_2(\mathbf{R})$, and $V_3(\mathbf{R})$ because the class of natural numbers is a natural generalization of the numbers one, two, three, and the class of fields is a natural generalization of the structure of the reals.

The second application of (PS) uses the above process of generalization to generate a very large class of spatiotemporal structures. This application is no more controversial, I think, than the first: one switches from Riemann's generalization of Euclidean space to Cantor's generalization of natural number. Thus consider the question: is there, for any ordinal number □, a possible spacetime with □ distinct instants of time? (PS) demands that the answer be "yes," supposing the possibility of discrete time. More exactly, I suppose that the following spacetime is possible: there are exactly □ instants of time, a first instant, a second instant, and so on for each natural number; space is Euclidean and three-dimensional, the structure $\mathbf{E}^3$; and spacetime is Newtonian. The spatiotemporal structure in question is thus the Cartesian product space, □x$\mathbf{E}^3$. Since □x$\mathbf{E}^3$ is possible, so are its substructures, nx$\mathbf{E}^3$, with n finite; call the class of such substructures Ω. Since the spatiotemporal structures in Ω are Newtonian, the temporal and spatial parameters can be independently varied. Now, apply the process of generalization illustrated above. Generate a larger class of spatiotemporal structures, Ω', by allowing the temporal parameter to range over the class of ordinal numbers, and the spatial parameter to range over any natural class of spatial structures containing $\mathbf{E}^3$. Ω' will be a natural generalization of Ω if the class of ordinal numbers is a natural generalization of the natural numbers. Which it is: the naturalness of Cantor's generalization of the natural numbers to the class of ordinals is adequately attested by the role the ordinals play in set theory and mathematics generally. So, by (PS), every member of Ω' is a possible structure; that is, for every ordinal number □, there is a possible spacetime with □ distinct instants of time.

Thus, if one accepts (PS), and the possibility of discrete time, one is lead to the conclusion that there is no set of possible worlds, that there are at least "as many" possible worlds as there are ordinal numbers.[16] According to David Lewis, this conclusion leads to trouble.[17] Lewis hopes to avoid the trouble by requiring that there be "some restriction on the possible size of spacetime." Against the charge, made by Peter Forrest and D. M. Armstrong,[18] that any such restriction is ad hoc, he replies:

> "Maybe so; the least arbitrary restriction we could possibly imagine is none at all, and compared to that any restriction whatever will seem at least somewhat ad hoc. But some will

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[16]This doesn't quite follow from (PS) alone: if there is a single world whose instants of time are similar in structure to the ordinal numbers in toto, then all members of Ω' may be instantiated at that one world. It does follow, however, if one assumes that, for any possible spacetime, there is a world at which that spacetime, and none larger, is instantiated. I won't argue for that here.

[17]To some extent, I agree: if there is no set of all worlds, then we cannot freely make use of set-theoretic constructions of worlds for purposes of semantics, epistemology, logic, whatever. But we can learn to live with the trouble, just as we have learned to live with the fact that there is no set of all sets. I propose an iterative conception of worlds according to which each world is assigned to a level of a cumulative hierarchy analogous to the hierarchy of pure sets; a plurality of worlds forms a set if and only if there is some level of the hierarchy that contains them all. There is no space to elaborate here.

[18]In "An Argument Against David Lewis' Theory of Possible Worlds," Australasian Journal of Philosophy, 62 (June, 1984), pp. 164-8.

> seem worse than others. A restriction to four-dimensional, or to seventeen-dimensional, manifolds looks badly arbitrary; a restriction to finite-dimensional manifolds looks much more tolerable. Maybe that is too much of a restriction, and disqualifies some shapes and sizes of spacetime that we would firmly believe to be possible. If so, then I hope there is some equally natural break a bit higher up: high enough to make room for all the possibilities we really need to believe in, but enough of a natural break to make it not intolerably _ad hoc_ as a boundary.
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> ... My thesis is existential: there is _some_ break, and the correct break is sufficiently salient within the mathematical universe not to be _ad hoc_. If study of the mathematical generalisations of ordinary spacetime manifolds revealed one salient break, and one only, I would dare to say that it was the right break--that there were worlds with all the shapes and sizes of spacetime below it, and no worlds with any other shapes and sizes. If study revealed no suitable breaks, I would regard that as serious trouble. If study revealed more than one suitable break, I would be content to profess ignorance.... [19]

It is clear from this passage that, in important respects, Lewis and I are in substantial agreement. We both require that the boundary marking the possible spacetimes be natural, and that its naturalness be decided by a study of the mathematical generalizations of ordinary spacetimes. But

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[19] _On the Plurality of Worlds,_ p. 103.

Lewis does not accept the full strength of (PS). Indeed, when Lewis' talk about natural breaks is translated into the terminology of natural classes, it appears that he would accept only some weaker principle such as (P2) (or (P3)). For Lewis, plenitude of spacetimes requires only that there be some natural break, that is, that the class of possible spacetimes be a natural class (or a union of natural classes).[20] He counts it no violation of plenitude if spacetimes belonging to more inclusive natural classes are deemed impossible.

I have already considered this view above; but let me reiterate one of my arguments in the present context. I think Lewis and I agree that plenitude demands that the possible spacetimes be characterizible in a non-arbitrary way. Lewis seems to think, however, that arbitrariness can be avoided as long as the boundary between the possible and the impossible spacetimes is a natural one. I disagree. Consider again the spacetimes in Ω'.[21] There are many natural breaks in the succession of ordinal numbers, for example, the first infinite ordinal, the first strong limit ordinal, the first inaccessible ordinal; and it does not seem plausible

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[20]With the proviso that it include those spacetimes whose possibility we "firmly" or "really need" to believe in. But what is the source of these firm beliefs? Not (B), apparently, for these beliefs may pertain to spacetimes of higher dimension. In any case, the proviso is no help in deciding between (PS) and a weaker principle; I have argued that we need to believe in the possibility of all the spacetimes warranted by (PS).

[21]The situation is less clear with respect to Lewis' example: the mathematical generalizations of continuous spacetimes to very large dimensions. I prefer to rest my argument on the spacetimes with discrete time, where the mathematics is trivial.

to hold that one such break is somehow more natural than any of the others. Each natural break in the succession of ordinals corresponds to a natural break among the spacetimes in Ω'. To allow any one such break to set a boundary in logical space, rather than any other, would be to allow arbitrariness in the characterization of logical space. That is why only a strong principle such as (PS) can successfully capture plenitude of structures.

VII

I conclude by summarizing the implications of my account for the epistemology of modality. I have attempted to steer a course between the Scylla of modal skepticism and the Charybdis of an obscurantist modal epistemology. The skeptic I have in mind holds that our only grounds for belief in the possibility of structures are the theoretical and explanatory grounds embodied in (B). Such skepticism is belied by ordinary practice, by our ordinary ways of thinking about modality. I take it our role as philosophers is not to challenge ordinary practice--except perhaps in rare cases--but to attempt to account for ordinary practice in a systematic way. I have developed and defended a principle of plenitude, (PS), that I think adequately explains and locates the source of our belief in a plenitude of possible structures. It warrants belief in the possibility of some structures that are not ordinarily thought to be possible; but these structures are not ordinarily thought to be possible, I think, only because they are not ordinarily thought of at all. My account does not purport to eliminate all ignorance as to which structures are logically possible. If a structure is not warranted by (PS) together with (B), it may or may not be logically possible, for all I have said; an absence of warranted belief does not warrant belief in the contrary.[22] Moreover, there is ignorance associated with the application of (B) and (PS): ignorance as to which structures are explanatorily adequate to actual phenomena translates into ignorance as to which structures are possible; as does ignorance as to the mathematical generalizations of structures. But although it may sometimes be unclear how my account applies in a particular case, the general grounds of our beliefs are made clear. When we infer that some structure is possible using (B) and (PS), we are guided by science (broadly construed) and by mathematics, not by some mysterious faculty of modal intuition. Nor is any such faculty needed to motivate or defend (B) and (PS). They are motivated and defended, not by modal intuition, but by what we require a theory of modality to do. And there need be nothing obscure about that.

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[22]My account is compatible with the view that <u>only</u> the structures warranted by (B) and (PS) are possible. But I would reject that view on grounds of parochialism; it would allow features of <u>our</u> inquiry, contingent and accidental though they be, to delimit the scope of the possible.